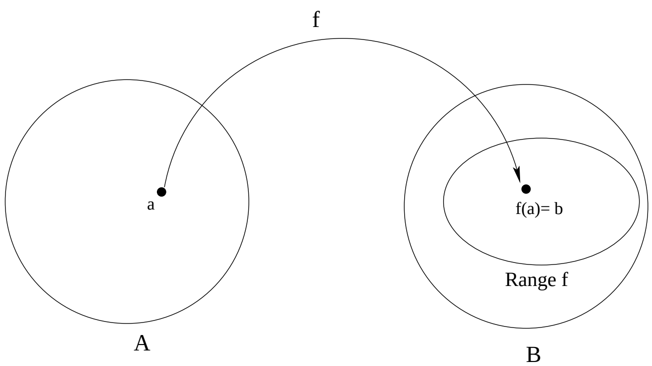
Math 310
Functions Handout

Definitions:

- Let A and B be non-empty sets. A **function** f from A to B, denoted $f : A \to B$ is an assignment of exactly one element of B to each element of A. We write $f(a) = b$ when b is the unique element of B that is assigned by f to the element a in A, and we say that the function f maps the set A into the set B.
- Continuing to use the notation defined above, we say that A is the **domain** of the function f and B is the **codomain** of f. The **range** of f is the set of all $b \in B$ such that there is an $a \in A$ with $f(a) = b$.
- If $f(a) = b$, then we call b the **image** of a. Using this language, we can say that the range of f is the union of all the images of the elements $a \in A$. Similarly, when $f(a) = b$, we say that a is an element of the **preimage** of b. The preimage of an element $b \in B$ is the set of all $a \in A$ such that $f(a) = b$. Notice that if b is not in the range, then its preimage is the empty set.



- Given a function $f : A \to B$, let $S \subseteq A$. The **image** of S under f is the subset of B that consists of the images of the elements of S. We denote the image of S by $f(S)$. Then $f(S) = \{t : \exists s \in S(t = f(s))\}$.

II. Graphs of Functions

Definition: Let $f : A \to B$ be a function. The **graph** of the function f is the set of all ordered pairs $\{(a, b) : a \in A$ and $f(a) = b$ $\}$.

Definition: A function f is **one-to-one** or **injective** if and only if either of the following equivalent conditions is satisfied:
(1) Whenever $a_1 \neq a_2$, $f(a_1) \neq f(a_2)$.
(2) Whenever $f(a_1) = f(a_2)$, $a_1 = a_2$.

Definition: A function f is **onto** or **surjective** if and only if for every $b \in B$, there is an element $a \in A$ with $f(a) = b$. That is, the codomain of f is equal to the range of f.

Notes:

- Recall that the **composition** of two functions $f : B \to C$ and $g : A \to B$, denoted by $f \circ g : A \to C$, is the function given by $f(g(a))$ for each $a \in A$.
- When a function f is one-to-one, we can define the **inverse function** f^{-1} as follows: $f^{-1}(b) = a$ when $f(a) = b$. Notice that, using this definition, $f \circ f^{-1}$ is the identity function $i_B : B \to B$ given by $i(b) = b$ for all $b \in B$, and $f^{-1} \circ f$ is the identity function $i_A : A \to A$ given by $i(a) = a$ for all $a \in A$.
- If a function f is both one-to-one and onto, then we say that f is a **bijection**.

Still More Definitions: Let $f : A \to B$ with $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$.

- We say f is **increasing** if $f(x) \leq f(y)$ whenever $x < y$ and $x, y \in A$.
- We say f is **strictly increasing** if $f(x) < f(y)$ whenever $x < y$ and $x, y \in A$.
- We say f is **decreasing** if $f(x) \geq f(y)$ whenever $x < y$ and $x, y \in A$.
- We say f is **strictly decreasing** if $f(x) > f(y)$ whenever $x < y$ and $x, y \in A$.