Math 450

Section 3.4: Hermite Interpolation

Main Idea: The Lagrange interpolating polynomial, $P_n(x)$, has been defined so that the polynomial agrees with the original function $f(x)$ at $n + 1$ distinct input values $x_0, x_1, \ldots, x_n$. On the other hand, Taylor polynomials approximate a function using a single center point at which we know the value of the function and several derivatives. Our goal is to generalize both the Lagrange polynomial and the Taylor polynomial by forming an interpolating polynomial that agrees with the function both at several distinct points and at a given number of derivatives of the function at those distinct points. A polynomial that satisfies these conditions is called an osculating polynomial.

Definition: Assume $x_0, x_1, \ldots, x_n \in [a, b]$ are $n + 1$ distinct numbers. Also, assume $m_0, m_1, \ldots, m_n$ are nonnegative integers where each integer, $m_i$, corresponds with $x_i$. Further assume $f \in C^m[a, b]$ where $m = \max\{m_i : 0 \leq i \leq n\}$. The osculating polynomial that approximates $f$ is the polynomial $P(x)$ of least degree such that $\frac{d^k P(x_i)}{dx^k} = \frac{d^k f(x_i)}{dx^k}$ for each $i = 0, 1, \ldots, n$ and $k = 0, 1, \ldots, m_i$. That is:

\[
\begin{align*}
f(x_0) &= P(x_0), f'(x_0) = P'(x_0), f''(x_0) = P''(x_0), \ldots, f^{(m_0)}(x_0) = P^{(m_0)}(x_0), \\
f(x_1) &= P(x_1), f'(x_1) = P'(x_1), f''(x_1) = P''(x_1), \ldots, f^{(m_1)}(x_1) = P^{(m_1)}(x_1), \\
&\quad \vdots \\
f(x_n) &= P(x_n), f'(x_n) = P'(x_n), f''(x_n) = P''(x_n), \ldots, f^{(m_n)}(x_n) = P^{(m_n)}(x_n).
\end{align*}
\]

Note: When there is a single point, $x_0$, the osculating polynomial approximating $f$ is the Taylor polynomial of $m_0$th degree.

Definition: The osculating polynomial of $f$ formed when $m_0 = m_1 = \cdots = m_n = 1$ is called the Hermite polynomial.

Note: The graph of the Hermite polynomial of $f$ agrees with $f$ at $n + 1$ distinct points and has the same tangent lines as $f$ at those $n + 1$ distinct points.

Recall: The Lagrange coefficient polynomial is defined by:

\[
L_{n,k} = \frac{(x - x_0) \cdots (x - x_{k-1})(x - x_{k+1}) \cdots (x - x_n)}{(x_k - x_0) \cdots (x_k - x_{k-1})(x_k - x_{k+1}) \cdots (x_k - x_n)}
\]

Also, $L_{n,k}(x_i) = \begin{cases} 0 & \text{if } i \neq k \\ 1 & \text{if } i = k \end{cases}$

Theorem: Assume $f \in C^1[a, b]$ and $x_0, x_1, \ldots, x_n \in [a, b]$ are distinct points. Then the unique polynomial of degree less than or equal to $2n + 1$ is given by:

\[
H_{2n+1}(x) = \sum_{j=0}^{n} f(x_j) H_{n,j}(x) + \sum_{j=0}^{n} f'(x_j) \hat{H}_{n,j}(x)
\]

where

\[
H_{n,j}(x) = [1 - 2(x - x_j)L'_{n,j}(x_j)] L^n_{n,j}(x)
\]

and

\[
\hat{H}_{n,j}(x) = (x - x_j)L^2_{n,j}(x)
\]
Example: Suppose that \( f(0) = 2, f'(0) = 1, f(1) = 4, f'(1) = -1, f(3) = 5, f'(3) = -2 \). Find the Hermite interpolating polynomial and use it to approximate the value of \( f(2) \).

Solution:

\[
L_{2,0}(x) = \frac{(x-1)(x-3)}{(0-1)(0-3)} = \frac{1}{3}x^2 - \frac{4}{3}x + 1. \text{ Therefore, } L'_{2,0}(x) = \frac{2}{3}x - \frac{4}{3}.
\]

\[
L_{2,1}(x) = \frac{(x-0)(x-3)}{(1-0)(1-3)} = -\frac{1}{2}x^2 + \frac{3}{2}x. \text{ Therefore, } L'_{2,1}(x) = -x + \frac{3}{2}.
\]

\[
L_{2,2}(x) = \frac{(x-0)(x-1)}{(3-0)(3-1)} = \frac{1}{6}x^2 - \frac{1}{6}x. \text{ Therefore, } L'_{2,2}(x) = \frac{1}{3}x - \frac{1}{6}.
\]

\[
H_{2,0}(x) = \left[ 1 - 2(x - 0)\left(\frac{2}{3}(0) - \frac{4}{3}\right) \right] \left[ \frac{1}{3}x^2 - \frac{4}{3}x + 1 \right] = (1 + \frac{8}{3}x) \left[ \frac{1}{3}x^2 - \frac{4}{3}x + 1 \right] - \frac{4}{3}x^2 + \frac{4}{3}x + 1 \right]^{2}
\]

\[
H_{2,1}(x) = \left[ 1 - 2(x - 1)\left(-1 + \frac{3}{2}\right) \right] \left[ -\frac{1}{2}x^2 + \frac{3}{2}x \right] = (2 - x) \left[ -\frac{1}{2}x^2 + \frac{3}{2}x \right] - \frac{4}{3}x^2 + \frac{4}{3}x + 1 \right]^{2}
\]

\[
H_{2,2}(x) = \left[ 1 - 2(x - 3)\left(\frac{1}{3}(3) - \frac{1}{6}\right) \right] \left[ \frac{1}{6}x^2 - \frac{1}{6}x \right] = (6 - \frac{5}{3}x) \left[ \frac{1}{6}x^2 - \frac{1}{6}x \right] - \frac{4}{3}x^2 + \frac{4}{3}x + 1 \right]^{2}
\]

\[
H_{2,0}(x) = (x - 0) \left[ \frac{1}{3}x^2 - \frac{4}{3}x + 1 \right] - \frac{4}{3}x^2 + \frac{4}{3}x + 1 \right]^{2}
\]

\[
H_{2,1}(x) = (x - 1) \left[ -\frac{1}{2}x^2 + \frac{3}{2}x \right] - \frac{4}{3}x^2 + \frac{4}{3}x + 1 \right]^{2}
\]

\[
H_{2,2}(x) = (x - 3) \left[ \frac{1}{6}x^2 - \frac{1}{6}x \right] - \frac{4}{3}x^2 + \frac{4}{3}x + 1 \right]^{2}
\]

\[
H_{5}(x) = 2 \left[ 1 + \frac{8}{3}x \right] \left[ \frac{1}{3}x^2 - \frac{4}{3}x + 1 \right] - \frac{4}{3}x^2 + \frac{4}{3}x + 1 \right]^{2} + 4 (2 - x) \left[ -\frac{1}{2}x^2 + \frac{3}{2}x \right] - \frac{4}{3}x^2 + \frac{4}{3}x + 1 \right]^{2} + 5 (6 - \frac{5}{3}x) \left[ \frac{1}{6}x^2 - \frac{1}{6}x \right] - \frac{4}{3}x^2 + \frac{4}{3}x + 1 \right]^{2} - (x - 1) \left[ -\frac{1}{2}x^2 + \frac{3}{2}x \right] - 2(x - 3) \left[ \frac{1}{6}x^2 - \frac{1}{6}x \right] - \frac{4}{3}x^2 + \frac{4}{3}x + 1 \right]^{2}
\]

\[
H_{5}(x) = -\frac{5}{6}x^5 + \frac{71}{12}x^4 - \frac{40}{3}x^3 + \frac{37}{4}x^2 + x + 2
\]

Using this, \( f(2) \approx H_{5}(2) = \frac{7}{3} \)

Note: We also could have used a divided difference table incorporating repeated values and derivatives to find this polynomial: