Math 476
Quotient Group Examples

Example 1: Let $G = D_4$ and let $\mathcal{H} = \{I, R_{180}\}$. As you (hopefully) showed on your daily bonus problem, $H \triangleleft G$.

In fact, the following are the equivalence classes in G induced by the cosets of $\mathcal{H}$:

$\mathcal{H} = \{I, R_{180}\}$, $R_{90}\mathcal{H} = \{R_{90}, R_{270}\} = \mathcal{H}R_{90}$, $H\mathcal{H} = \{H, V\} = \mathcal{H}H$, and $D_1\mathcal{H} = \{D_1, D_2\} = \mathcal{H}D_1$

Let's start by rearranging the rows and columns of the Cayley Table of D_4 so that elements in the same equivalence class are next to each other:

	I	R_{180}	R_{90}	R_{270}	H	V	D_1	D_2
I	I	R_{180}	R_{90}	R_{270}	H	V	D_1	D_2
R_{180}	R_{180}	I	R_{270}	R_{90}	V	H	D_2	D_1
R_{90}	R_{90}	R_{270}	R_{180}	I	D_2	D_1	H	V
R_{270}	R_{270}	R_{90}	I	R_{180}	D_1	D_2	V	H
H	H	V	D_1	D_2	I	R_{180}	R_{90}	R_{270}
V	V	H	D_2	D_1	R_{180}	I	R_{270}	R_{90}
D_1	D_1	D_2	V	H	R_{270}	R_{90}	I	R_{180}
D_2	D_2	D_1	H	V	R_{90}	R_{270}	R_{180}	I

Next, we build the Cayley Table for the Quotient group G/H and compare:

	$\mathcal{H}$	$R_{90}\mathcal{H}$	$H\mathcal{H}$	$D_1\mathcal{H}$
$\mathcal{H}$	$\mathcal{H}$	$R_{90}\mathcal{H}$	$H\mathcal{H}$	$D_1\mathcal{H}$
$R_{90}\mathcal{H}$	$R_{90}\mathcal{H}$	$\mathcal{H}$	$D_1\mathcal{H}$	$H\mathcal{H}$
$H\mathcal{H}$	$H\mathcal{H}$	$D_1\mathcal{H}$	$\mathcal{H}$	$R_{90}\mathcal{H}$
$D_1\mathcal{H}$	$D_1\mathcal{H}$	$H\mathcal{H}$	$R_{90}\mathcal{H}$	$\mathcal{H}$

Notice that if we think of collapsing each 2×2 subsection of the first Cayley Table into a single square labelled by the related coset, we obtain the second Cayley Table. Perhaps the best way to think of the process of creating a Quotient group using a normal subgroup is that we are using the partition formed by the collection of cosets to define an equivalence relation of the original group G. We make this into a group by defining coset "multiplication". This results in a group precisely when the subgroup H is normal in G.

A final question to address is this: what happens if we attempt this same process with a subgroup that it not normal? Let's attempt this with $\mathcal{H} = \{I, V\}$. Since $R_{90}\mathcal{H} = \{R_{90}, D_1\}$, while $\mathcal{H}R_{90} = \{R_{90}, D_2\}$, H is not normal in G.

The following are the left cosets of H in G:

$\mathcal{H} = \{I, V\}$, $R_{90}\mathcal{H} = \{R_{90}, D_1\}$, $R_{180}\mathcal{H} = \{R_{180}, H\}$, and $R_{270}\mathcal{H} = \{R_{270}, D_2\}$

Let's rearranging the rows and columns of the Cayley Table of D_4 so that elements in the same left coset are next to each other:

	I	V	R_{90}	D_1	R_{180}	H	R_{270}	D_2
I	I	V	R_{90}	D_1	R_{180}	H	R_{270}	D_2
V	V	I	D_2	R_{270}	H	R_{180}	D_1	R_{90}

We'll stop here as we can already see that we have insurmountable problems.