Math 476
Exam 1

Instructions: You will have 55 minutes to complete this exam. The credit given on each problem will be proportional to the amount of correct work shown. Answers without supporting work will receive little credit. Work your exam on separate sheets of paper. Be sure to number each problem and put your name on each page.

1. Consider the following operation on $\mathbb{R}$: $a \star b = a + \frac{b}{2}$

   (a) (5 points) Determine, with proof, whether or not the operation $\star$ is commutative.

   This operation is not commutative. To see this, let $a = 1$ and $b = 2$. Then $a \star b = 1 + \frac{2}{2} = 1 + 1 = 2$ while $b \star a = 2 + \frac{1}{2} = \frac{5}{2}$.

   (b) (5 points) Determine, with proof, whether or not the operation $\star$ is associative.

   This operation is not associative. Let $a = 1$, $b = 2$, and $c = 3$. First, $a \star (b \star c) = 1 \star (2 \star 3) = 1 \star \left(2 + \frac{3}{2}\right) = 1 \star \left(\frac{7}{2}\right) = 1 + \frac{7}{4} = \frac{11}{4}$.

   On the other hand, $(a \star b) \star c = (1 \star 2) \star 3 = (1 + 1) \star 3 = 2 \star 3 = 2 + \frac{3}{2} = \frac{7}{2}$.

   (c) (5 points) Determine whether or not there is an identity element for the operation $\star$.

   Suppose $a \star b = a$. That $a + \frac{b}{2} = a$, so $\frac{b}{2=0}$. Therefore, $b = 0$ acts as a right identity.

   However, we also need $b \star a = a$, in which case, $b + \frac{a}{2} = a$, so $b = \frac{a}{2}$. Since no constant value satisfies this equation, there is no right inverse, hence there is no inverse element in $\mathbb{R}$ under the operation $\star$.

   (d) (5 points) Determine which elements of $\mathbb{R}$ have inverses under $\star$, and find a formula expressing the inverse of an element a for those elements that have an inverse.

   Since there is no inverse element, no elements have an inverse element.

2. (6 points) Make a Cayley Table for the group $U(8)$.

   Recall that the elements of $U(8)$ are those in $\mathbb{Z}_8$ that are relatively prime with 8. Then $U(8) = \{1, 3, 5, 7\}$. The Cayley Table for this group is:

	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

3. (10 points) Prove **one** of the following Theorems:

   (a) **Theorem 2.1** In a group G, there is only one identity element.

   Proof:

   First note that since G is a group, by definition, there is an identity element e such that for all $g \in G$, $eg = ge = g$.

   Suppose that f is also an identity element. Then for all $g \in G$, $fg = gf = g$.

   Consider ef. Since e is a left identity, $ef = f$. However, since f is a right identity, $ef = e$. Hence $e = f$. $\square$.

(b) **Theorem 2.3** For each element a in a group G, there is a unique element b in G such that $ab = ba = e$ [Note: you may assume that the left and right cancellation laws hold in a group for this proof]

Proof:

Let $a \in G$ and suppose that b and c are both inverses of a. Then $ab = e$ and $ac = e$. Therefore, $ac = b$. Also, $ba = e$ and $ca = e$.

Notice that $b(ac) = be = b$. However, since G is associative, $b(ac) = (ba)c = ec = c$. Hence $b = c$. $\square$.

4. Consider the Group given by the following Cayley Table:

$\star$	e	a	b	c	d	f
e	e	a	b	c	d	f
a	a	b	f	e	c	d
b	b	f	d	a	e	c
c	c	e	a	d	f	b
d	d	c	e	f	b	a
f	f	d	c	b	a	e

(a) (5 points) Is this group Abelian? Justify your answer.

Yes. Notice that the Cayley Table for this group is symmetric along the main diagonal. Therefore, one can see that $a \star b = b \star a$ for every pair of elements in this group.

(b) (5 points) Find the inverse of the element d.

Notice that e acts as the identity element in this group. Therefore, since $d \star b = b \star d = e$, $d^{-1} = b$.

(c) (5 points) Find the order of the element a.

Notice that $a^2 = b$, $a \star b = f$, $a \star f = d$, $a \star d = c$, and $a \star c = e$. Since $|a| = 6$ [In fact, a is a cyclic generator for $U(8)$.]

5. (10 points) Let H and K be subgroups of a group G. Prove that $H \cap K$ is a subgroup of G.

Proof:

Notice that since $e \in H$ and $e \in K$, then $H \cap K$ is non-empty. We will proceed by using the two step subgroup test.

Suppose $a, b \in H \cap K$. Since H is a subgroup and $a, b \in H$, then $ab \in H$. Similarly, Since K is a subgroup and $a, b \in K$, then $ab \in K$. Therefore, $ab \in H \cap K$, so $H \cap K$ is closed under the group operation.

Next, suppose $a \in H \cap K$. Since H is a subgroup and $a \in H$, then $a^{-1} \in H$. Similarly, Since K is a subgroup and $a \in K$, then $a^{-1} \in K$. Therefore, $a^{-1} \in H \cap K$, so $H \cap K$ is closed under taking inverses. Therefore, $H \cap K \leq G$. $\square$.

6. Consider the group $G = \mathbb{Z}_{18}$.

(a) (5 points) Give a complete list of the elements of G that are cyclic generators for the entire group.

Recall that by Corollary 3 to Theorem 4.2, an element k is a generator of $\mathbb{Z}_{18}$ is and only if $gcd(k, 18) = 1$.

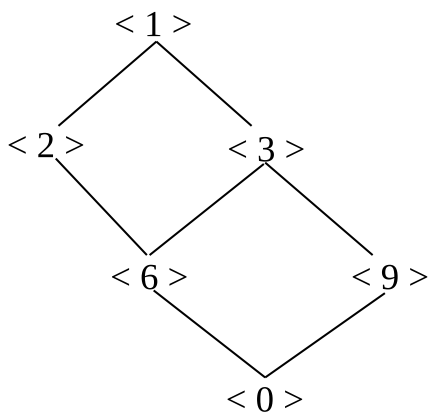
Then the following is a complete list of generators for $\mathbb{Z}_{18}$: $\{1, 5, 7, 11, 13, 17\}$

(b) (5 points) Give a complete list of the elements of G that generate a subgroup of order 6.

Recall that by Corollary 2 to Theorem 4.2, $\langle a^i \rangle = \langle a^j \rangle$ if and only if $gcd(n,i) = gcd(n,j)$. Also, by Theorem 4.3, the unique subgroup of order k in a cyclic group of order n is $\langle a^{\frac{n}{k}} \rangle$.

Then $\langle 3 \rangle$ is the order 6 subgroup of $\mathbb{Z}_{18}$. It is generated by any element $k \in \mathbb{Z}_{18}$ such that $gcd(n,k) = 3$. The following is a complete list of the generators: $3, 15$.

(c) (8 points) Draw the subgroup lattice for this group.



7. Consider the permutation $\sigma = (1\,3\,7)(2\,4)(3\,5\,6)$

(a) (5 points) Write σ in disjoint cycle notation.

Carrying out multiplication (in composition order) gives: $(1\,3\,5\,6\,7)(2\,4)$

(b) (5 points) Determine whether σ is even or odd.

Recall that $(1\,3\,7) = (1\,7)(1\,3)$ and $(3\,5\,6) = (3\,6)(3\,5)$

Then $\sigma = (1\,7)(1\,3)(2\,4)(3\,6)(3\,5)$

Since σ can be expressed as the product of 5 2-cycles, σ is odd.

(c) (3 points) Find the order of σ

Since the disjoint cycle form of σ consists of a 5-cycle and a 2-cycle, then, by Theorem 5.3, $|\sigma| = 10$.

8. Consider the group S_6.

(a) (6 points) Use disjoint cycle structures to find a complete list of the orders of elements in S_6.

The list of possible cycle structures in S_6 and resulting order is as follows:

Cycle Form	Order
(6)	6
(5)(1)	5
(4)(2)	4
(4)(1)(1)	4
(3)(3)	3
(3)(2)(1)	6
(3)(1)(1)(1)	3
(2)(2)(2)	2
(2)(2)(1)(1)	2
(2)(1)(1)(1)(1)	2
(1)(1)(1)(1)(1)(1)	1

The possible orders for elements in S_6 are: 6, 5, 4, 3, 2, 1.

(b) (4 points) Find the order of A_6.

Recall that $|A_6| = \frac{6!}{2} = \frac{720}{2} = 360$.

(c) (5 points) Find the number of elements of S_6 that have order 5.

Notice that there are $P(6, 5) = 720$ ways of constructing a 5-cycle in S_6.

However, as we saw on the similar homework problem, these cycles are not all distinct. For example, the 5-cycle $(1\,2\,3\,4\,5)$ is equivalent to the cycle $(5\,1\,2\,3\,4)$. All in all, there are 5 distinct cycles that represent the same group element. Hence S_6 has $\frac{720}{5} = 144$ elements of order 5 in S_6.