Math 476
Exam 2

Instructions: You will have 55 minutes to complete this exam. The credit given on each problem will be proportional to the amount of correct work shown. Answers without supporting work will receive little credit.
Work your exam on separate sheets of paper. Be sure to number each problem and put your name on each page.

1. (10 points) Let $\varphi : G \to \overline{G}$ be an isomorphism. Prove **one** of the following:

   (a) $\varphi(e) = \overline{e}$. [$\varphi$ maps the identity in G to the identity in $\overline{G}$]

   Recall that in any group $e \cdot e = e$. Then, using the group operation preservation property of an isomorphism, $\varphi(e) = \varphi(e \cdot e) = \varphi(e)\varphi(e)$. That is, $\varphi(e) = \varphi(e)\varphi(e)$. Therefore, using the left cancellation property (or, left multiplying by $[\varphi(e)]^{-1}$), we have $\overline{e} = \varphi(e)$. $\square$.

   (b) If $H \leq G$, then $\varphi(H) \leq \overline{G}$ [φ maps subgroups to subgroups]

   First, note that since $e \in H$, $\varphi(e) \in \varphi(H)$, so $\varphi(H) \neq \emptyset$.

   Let $\overline{h_1}, \overline{h_2} \in \varphi H$. Then there exist elements $h_1, h_2 \in G$ such that $\varphi(h_1) = \overline{h_1}$, and $\varphi(h_2) = \overline{h_2}$. Since $H \leq G$, $h_2^{-1} \in H$ (subgroups are closed under taking inverses). Then $h_1 \cdot h_2^{-1} \in H$ as well (subgroups are closed under the group operation).

   By definition, $\varphi\left(h_1 \cdot h_2^{-1}\right) \in \varphi(H)$. However, since φ is an isomorphism, $\varphi\left(h_1 \cdot h_2^{-1}\right) = \varphi(h_1)\varphi\left(h_2^{-1}\right) = \varphi(h_1)\left[\varphi(h_2)\right]^{-1}$.

   Hence, using the one step subgroup test, $\varphi(H) \leq \overline{G}$.

2. (10 points) Let G be a group and H a subgroup of G. Prove that $aH = bH$ if and only if $a \in bH$.

   First, one should notice that there are two directions to prove.

   First, suppose that $aH = bH$. Since $e \in H$, then $a \cdot e = a \in aH$ ($aH = \{ah : h \in H\}$). Therefore, since $aH = bH$, $a \in bH$.

   Next, suppose that $a \in bH$. Then $a = bh$ for some $h \in H$. Recall that for any $h \in H$, $hH = H$, since for any $h' \in H$, $h^{-1}h' \in H$, so $h\left(h^{-1}h'\right) = h' \in hH$, and, since H is a subgroup, for any $h'' \in H$, $hh'' \in H$. Therefore, $aH = (bh)H = b(hH) = bH$.

3. (8 points) Let H and K be subgroups of a group G. Suppose $|H| = 24$ and $|K| = 18$. Find all possible values for $|H \cap K|$. Briefly justify your answer.

   First, recall that, as proven on a previous homework assignment, $HcapK$ is a subgroup of G. In fact, $(H \cap K) \leq H$ and $(H \cap K) \leq K$.

   Next, recall that by Lagrange's Theorem, the order of a subgroup always divides the order of a group containing that subgroup. From this, we know that $|H \cap K|$ divides both 18 and 24. Notice that the factors of 18 are: $1, 2, 3, 6, 9, 18$ while the factors of 24 are: $1, 2, 3, 4, 6, 8, 12, 24$. Then a complete list of the possible orders for $|H \cap K|$ is: $1, 2, 3, 6$.

4. (6 points) Explain why D_4 and $\mathbb{Z}_8$ are not isomorphic even though both are groups of order 8.

   There are quite a few ways to see that these two groups are not isomorphic. A few of you tried to argue that there is no bijection between the two, but that is not true. There are 8! bijections between these groups, 7! of which send the identity to the identity. The problem comes when trying to preserve the group operation. The main ways to show that two groups are not isomorphic is to assume that there is an isomorphism and demonstrate that this leads to a contradiction, or to show that a property that is preserved by all isomorphisms is not shared by both groups. I will take the second approach.

   Method 1: Recall that isomorphisms preserve abelian groups. That is, of G_1 and G_2 are isomorphic and G_1 is abelian, then G_2 is also abelian. However, D_4 is not abelian, while $\mathbb{Z}_8$ is abelian. Hence D_4 and $\mathbb{Z}_8$ are not isomorphic.

   Method 2: Recall that isomorphisms preserve the order of group elements. That is, if phi is an isomorphism between G_1 and G_2 and $g \in G_1$, then $|g| = |\phi(g)|$. However while $\mathbb{Z}_8$ has elements of order 8, D_4 does not have any elements of order 8. Hence D_4 and $\mathbb{Z}_8$ are not isomorphic.

5. Let $G = \mathbb{Z}_6 \oplus \mathbb{Z}_4$

   (a) (2 points) Find $|G|$.

   Since elements of G are ordered pairs drawn from $\mathbb{Z}_6$ and $\mathbb{Z}_4$ respectively, $|G| = 6 \cdot 4 = 24$.

   (b) (5 points) Find $|(2,2)|$

   The simplest way to see this is to recall that $|(g_1, g_2)| = lcm\left(|g_1|, |g_2|\right)$.

   Since $|g_1| = |2| = 3$ (here we are thinking of 2 in $\mathbb{Z}_6$) and $|g_2| = |2| = 2$ (here we are thinking of 2 in $\mathbb{Z}_4$), then $|(2,2)| = lcm(3,2) = 6$.

   One could also find this directly by taking $(2,2)$ to increasing powers.

   (c) (5 points) Is G cyclic? Justify your answer.

   No. Using the result $|(g_1, g_2)| = lcm\left(|g_1|, |g_2|\right)$, since the maximum order of elements in $\mathbb{Z}_6$ in 6, and the maximum order of elements in $\mathbb{Z}_4$ is 4, then the maximum order in the sum is $lcm(6,4) = 12$. Since this is less than the order of the group (24), there can be no cyclic generator, so $\mathbb{Z}_6 \oplus \mathbb{Z}_4$ is not a cyclic group.

   A much simpler way to solve this problem was to recall that we have a theorem that states that $\mathbb{Z}_m \oplus \mathbb{Z}_n \cong \mathbb{Z}_{mn}$ if and only if $gcd(m,n) = 1$. Here, $gcd(6,4) = 2 > 1$, so this group is not cyclic.

   (d) (6 points) Find all elements of G that have order 4.

   Notice that $\mathbb{Z}_6$ has no elements of order 4, since 4 does not divide 6. Therefore, to get an element of order 4, we must choose an element of order 4 from $\mathbb{Z}_4$ is go in the second coordinate, and we need an element of order 1 or 2 from $\mathbb{Z}_6$ to go into the first component.

   Notice that the only elements of order 1 and 2 in $\mathbb{Z}_6$ are 0 and 3 respectively. The elements of order 4 in $\mathbb{Z}_4$ are the generators 1 and 3. Therefore, the following is a complete list of elements of order 4 in G:

   $(0,1)$, $(0,3)$, $(3,1)$, $(3,3)$

6. Let $G = \mathbb{Z}_{12}$ and let $H = \langle 4 \rangle$

(a) (4 points) Find the left cosets of H in G.

First notice that $H = \langle 4 \rangle = \{0, 4, 8\}$.

Then the following is a complete list of the left cosets of H in G:

$0 + H = \{0, 4, 8\}$, $1 + H = \{1, 5, 9\}$, $2 + H = \{2, 6, 10\}$, and $3 + H = \{3, 7, 11\}$.

(b) (4 points) Prove that $H \triangleleft G$.

There were three main ways to approach this problem. The simplest was to observe that since G is abelian, **all** subgroups of G are normal.

The second method was to compute the right cosets and compare them pairwise to the left cosets.

$H + 0 = \{0, 4, 8\}$, $H + 1 = \{1, 5, 9\}$, $H + 2 = \{2, 6, 10\}$, and $H + 3 = \{3, 7, 11\}$.

Since the left and right cosets are pairwise equal ($a + H = H + a$ for all $a \in G$) then $H \triangleleft G$.

The third way to show normality was to use the normality test and show that for each $x \in G$, that $xHx^{-1} \subset H$ (in fact, since G is abelian, one can easily show that for all $x \in G$, $xHx^{-1} = H$).

(c) (8 points) Construct a Cayley Table for the factor group G/H and use it to determine whether or not G/H is cyclic.

Note that since the group $\mathbb{Z}_12$ is an additive group, one should use additive notation when writing the cosets. Here is a Cayley Table for the factor group G/H:

	$0+H$	$1+H$	$2+H$	$3+H$
$0+H$	$0+H$	$1+H$	$2+H$	$3+H$
$1+H$	$1+H$	$2+H$	$3+H$	$0+H$
$2+H$	$2+H$	$3+H$	$0+H$	$1+H$
$3+H$	$3+H$	$0+H$	$1+H$	$1+H$

Note that we used the operation $(a + H) + (b + H) = (a + b)H$, and if necessary, replaced the representative $(a + b)$ with the standard equivalent coset representative.

Many of you tried to use the fact that the Cayley Table is symmetric to conclude that this group is cyclic. This does not work. Having a symmetric Cayley Table means that the group is abelian, but there are abelian groups which are not cyclic (for example, $\mathbb{Z}_2 \oplus \mathbb{Z}_2$).

G/H is indeed cyclic. To show this, we note that $1 + H$ has order 4, and hence is a cyclic generator for the group.

7. Let $\psi : \mathbb{Z}_{15} \to \mathbb{Z}_{20}$ be given by $\psi(k) = 4k$.

   (a) (4 points) Explain why ψ must be a homomorphism.

   Notice that for any $a, b \in \mathbb{Z}_{15}$, $\psi(a+b) = 4(a+b) = 4a + 4b = \psi(a) + \psi(b)$ (where these are all computed modulo 20). Hence ψ is a homomorphism.

   (b) (4 points) Find the kernel of ψ.

   The kernel consists of all elements mapped to 0 in $\mathbb{Z}_{20}$. Since the maps acts by multiplying elements by 4, this means that the kernel is the set of elements that are multiples of 5. That is, $ker\psi = \{0, 5, 10\}$.

   (c) (4 points) Find the image of ψ.

   Since the generator 1 in $\mathbb{Z}_{15}$ is mapped to the element 4 in $\mathbb{Z}_{20}$, $image\psi = \langle 4 \rangle = \{0, 4, 8, 12, 16\}$.

   (d) (4 points) Find $\psi^{-1}(12)$.

   Since $\psi(3) = 12$, then $\psi^{-1}(12) = 3 + ker\psi = \{3, 8, 13\}$.

8. (10 points) List all isomorphism classes for an abelian group G if $|G| = 500$.

Using the Fundamental Theorem of Abelian Finite Abelian Groups and the Theorem that $\mathbb{Z}_m \oplus \mathbb{Z}_n \cong \mathbb{Z}_{mn}$ if and only if $gcd(m, n) = 1$, we can find all isomorphism classes using sums of groups of the form $\mathbb{Z}_n$ and by looking at "partitioning" the prime factors of 500.

Notice that $500 = 5^3 \cdot 2^2$. Therefore, the following is a complete list of the isomorphism classes of abelian groups of order 500:

(a) $\mathbb{Z}_{500} \cong \mathbb{Z}_{125} \oplus \mathbb{Z}_4$
(b) $\mathbb{Z}_{250} \oplus \mathbb{Z}_2 \cong \mathbb{Z}_{125} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$
(c) $\mathbb{Z}_{100} \oplus \mathbb{Z}_5 \cong \mathbb{Z}_{25} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_4$
(d) $\mathbb{Z}_{50} \oplus \mathbb{Z}_{10} \cong \mathbb{Z}_{25} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$
(e) $\mathbb{Z}_{20} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5 \cong \mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_4$
(f) $\mathbb{Z}_{10} \oplus \mathbb{Z}_{10} \oplus \mathbb{Z}_5 \cong \mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$