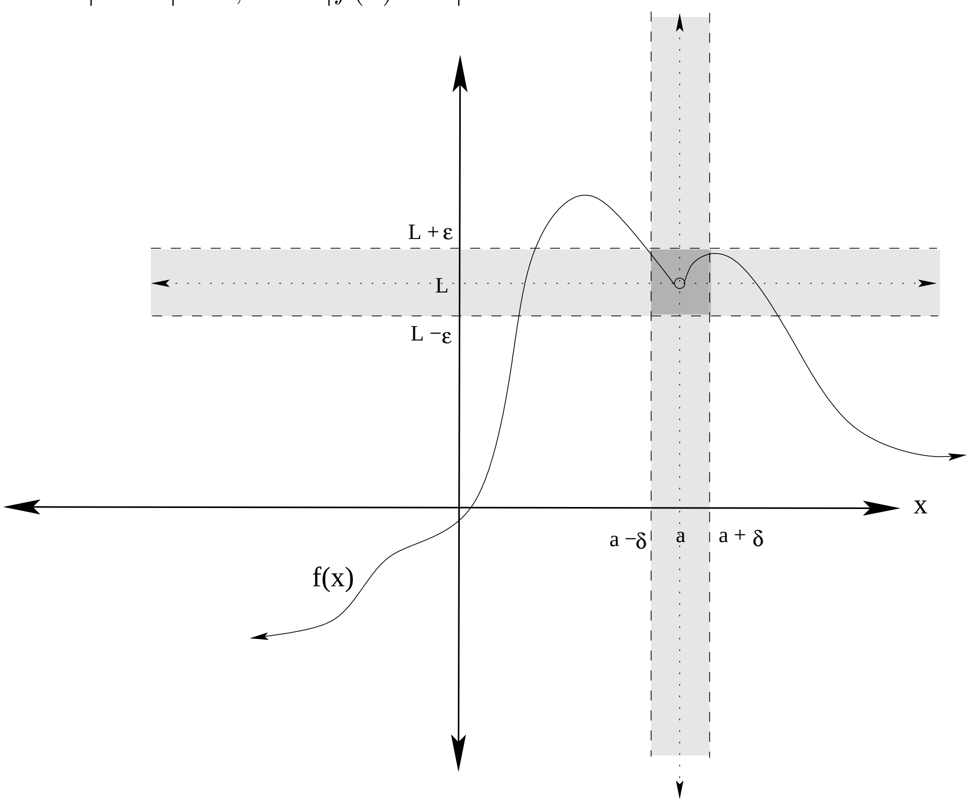
Math 261
The Definition of a Limit Handout

Definition: Let f be a function defined on an open interval containing a, except possibly at a itself, and let L be a real number. The statement $\lim_{x\to a} f(x) = L$ means that for every $\epsilon > 0$ there is a $\delta > 0$ such that if $0 < |x-a| < \delta$, then $|f(x) - L| < \epsilon$.



Example 1: Let $f(x) = \sqrt[3]{x-1}$ and let $a = 9$ and $L = 2$. Let $\epsilon = .001$. Find $\delta > 0$ such that if $0 < |x-a| < \delta$, then $|f(x) - L| < \epsilon$.
Solution: We need $|\sqrt[3]{x-1} - 2| < .001$. That is, $-.001 < \sqrt[3]{x-1} - 2 < .001$ or $1.999 < \sqrt[3]{x-1} < 2.001$. Cubing both sides, we obtain: $7.988006 < x - 1 < 8.012006$, or $8.988006 < x < 9.012006$.
Notice $9 - 8.988006 = .011994$, while $9.012006 - 9 = .012006$. Therefore, taking $\delta = .01$ will suffice. That is, if $0 < |x-9| < .01$ (or $8.99 < x < 9.01$), then $|\sqrt[3]{x-1} - 2| < .001$

Example 2: Use the definition of the limit of a function to prove that $\lim_{x\to 2} 5 - 2x = 1$

Let $\epsilon > 0$. Suppose that $|f(x) - L| < \epsilon$.
Then $|5 - 2x - 1| = |4 - 2x| < \epsilon$.
But then $2|2 - x| = 2|x - 2| < \epsilon$, so $|x - 2| < \frac{\epsilon}{2}$.

Therefore, let $\delta \le \frac{\epsilon}{2}$, and suppose $0 < |x-2| < \delta$.
Then $2|x-2| = 2|2-x| < 2\delta \le \epsilon$.
Therefore $|4 - 2x| = |5 - 2x - 1| < \epsilon$, or $|f(x) - 1| < \epsilon$.

Thus $\lim_{x\to 2} 5 - 2x = 1$