Math 261

Numerical Integration Handout

Recall: When we first defined the definite integral, we did so using the idea of a **Riemann sum**. Each Riemann sum corresponds to approximating the area between a function and the x-axis using rectangles based on a partition P and some choice of input values in order to compute the height of each rectangle. We used Left Endpoints, Right Endpoints, Midpoints, Upper Sums, and Lower sums over equally spaced partitions in order to approximate the value of several definite integrals.

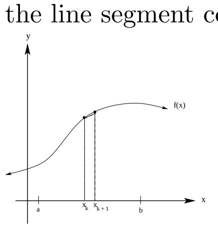
A downside of these methods for approximating the value of a definite integral is that they were not particularly accurate. Our goal for today is to find more effective ways of approximating the value of a definite integral. One approach that was mentioned in passing earlier is to use trapezoids rather than rectangles to approximate the area under our function.

The Trapezoid Rule:

In this method, we approximate the value of $\int_a^b f(x)\,dx$ as follows:

Think of subdividing $[a,b]$ into n equal subintervals. Each subinterval has width $\Delta x = \frac{b-a}{n}$.

Next, look at the area of the trapezoid formed by drawing in vertical lines at each point in the partition, and then drawing the line segment connecting the points on the graph of the function at the left and right endpoints of a subinterval:



Recall that the area of a trapezoid with heights h_1 and h_2 and width w is given by $A = \frac{1}{2}(h_1+h_2)\cdot w$. Then the area of this trapezoid is: $A_k = \frac{1}{2}(f(x_k)+f(x_{k+1}))\cdot \Delta x = \frac{1}{2}(f(x_k)+f(x_{k+1}))\cdot \frac{b-a}{n} = \frac{b-a}{2n}(f(x_k)+f(x_{k+1}))$.

From this, an approximation of the area of the entire region can be found by adding up the area of all trapezoids of this form. That is:

$$A = \int_a^b f(x)\,dx \approx \frac{b-a}{2n}(f(x_0)+f(x_1)) + \frac{b-a}{2n}(f(x_1)+f(x_2)) + \frac{b-a}{2n}(f(x_2)+f(x_3)) + \cdots + \frac{b-a}{2n}(f(x_{n-2})+f(x_{n-1})) + \frac{b-a}{2n}(f(x_{n-1})+f(x_n))$$

$$= \frac{b-a}{2n}\left((f(x_0)+f(x_1)) + (f(x_1)+f(x_2)) + (f(x_2)+f(x_3)) + \cdots + (f(x_{n-2})+f(x_{n-1})) + (f(x_{n-1})+f(x_n))\right)$$

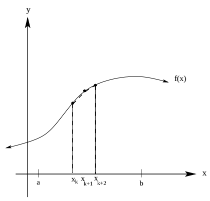
Thus $A = \int_a^b f(x)\,dx \approx \frac{b-a}{2n}(f(x_0) + 2\cdot f(x_1) + 2\cdot f(x_2) + \cdots + 2\cdot f(x_{n-1}) + f(x_n))$

Example: Let $f(x) = x^2+4$. Use the Trapezoid Rule with $n=6$ to approximate the value of $\int_2^5 f(x)\,dx$.

Simpson's Rule:

If we think carefully about the approximation methods that we have used so far, we went from using rectangles to approximate area, which corresponds to using constant functions to fit $f(x)$, to using trapezoids, which corresponds to using linear functions to fit $f(x)$.

The next natural thing to try is to fit $f(x)$ using quadratic functions.



In the interest of time, I refer you to your textbook for the derivation, but the area of the parabolic section given by the points $(x_k, f(x_k))$, $(x_{k+1}, f(x_{k+1}))$, and $(x_{k+2}, f(x_{k+2}))$ is: $A_k = \frac{1}{3}\left(f(x_k) + 4 \cdot f(x_{k+1}) + f(x_{k+2}\right) \cdot \Delta x = \frac{1}{3}\left(f(x_k) + 4 \cdot f(x_{k+1}) + f(x_{k+2})\right) \cdot \frac{b-a}{n} = \frac{b-a}{3n}\left(f(x_k) + 4 \cdot f(x_{k+1}) + f(x_{k+2})\right)$.

From this, an approximation of the area of the entire region using parabolic sectors is:

$$A = \int_a^b f(x)\,dx \approx \frac{b-a}{3n}\left(f(x_0) + 4f(x_1) + f(x_2)\right) + \frac{b-a}{3n}\left(f(x_2) + 4f(x_3) + f(x_4)\right) + \cdots + \frac{b-a}{3n}\left(f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right)$$

Thus $A = \displaystyle\int_a^b f(x)\ dx \approx \frac{b-a}{3n}\left(f(x_0) + 4 \cdot f(x_1) + 2 \cdot f(x_2) + 4f(x_3) + 2f(x_4) \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right)$. Note that n must be even for this formula to apply.

Example: Let $f(x) = x^2 + 4$. Use Simpson's Rule with $n = 6$ to approximate the value of $\displaystyle\int_2^5 f(x)\,dx$.

Error Formulas:

When we use approximation methods like the Trapezoid Rule and Simpson's Rule, one important issue is whether or not we can figure out how accurate our results are to the true answer. In practice, the best we can hope for (unless we take the time to compute the actual answer, which makes approximating silly and may not even be possible) is to find an upper bound on the amount of error in a given approximation. We will not take the time to prove the following error formulas (the proofs require theory that you would not learn until the very end of Calculus 2), but here are error bounds for these two approximation methods:

Trapezoid Rule Error: Suppose that $f''(x)$ is continuous on $[a, b]$ and that M is a positive number so that $|f''(x)| \leq M$ for every x in $[a, b]$. Then the error E_n in using the Trapezoid Rule with n equally spaced subdivisions to approximate $\displaystyle\int_a^b f(x)\,dx$ satisfies

$E_n \leq \frac{M(b-a)^3}{12n^2}$

Simpson's Rule Error: Suppose that $f^{(4)}(x)$ is continuous on $[a, b]$ and that M is a positive number so that $|f^{(4)}(x)| \leq M$ for every x in $[a, b]$. Then the error E_n in using Simpson's Rule with n equally spaced subdivisions to approximate $\displaystyle\int_a^b f(x)\,dx$ satisfies

$E_n \leq \frac{M(b-a)^5}{180n^4}$