Math 261
Continuity

Definition: A function f is **continuous** at an interior point c of its domain if:

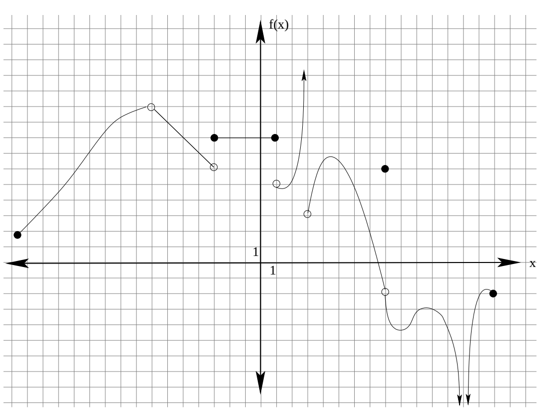
- $f(c)$ is defined
- $\lim_{x \to c} f(x)$ exists
- $\lim_{x \to c} f(x) = f(c)$

A function f is **continuous** at a left endpoint a or a right endpoint b of its domain if:

- $f(a)$ is defined, $\lim_{x \to a^+} f(x)$ exists, and $\lim_{x \to a^+} f(x) = f(a)$.
  or
- $f(b)$ is defined, $\lim_{x \to b^-} f(x)$ exists, and $\lim_{x \to b^-} f(x) = f(b)$.

Notes:

- We really only need to check the third condition, since the first two are implied by the third.
- If one (or more) of the three conditions in the definition above are not satisfied, we say that f is **discontinuous** at the corresponding value or that f has a **discontinuity** at that value.
- We can extend the definition above to say that a function $f(x)$ is continuous on an open interval (a, b) if it is continuous at every interior point if this interval. We can further extend it to say that a function $f(x)$ is continuous on a closed interval $[a, b]$ is it is continuous at every interior point and is continuous at both endpoints as well.



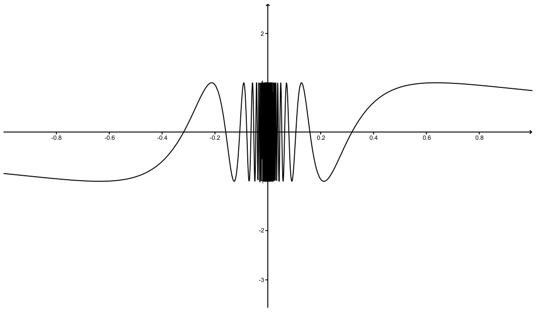
We often classify discontinuities as one of four main types.

1. If $\lim_{x \to c} f(x)$ exists but $f(c)$ is undefined or attains a different value, we say that f has a **removable discontinuity** at c. This is due to the fact that redefining the value of $f(c)$ would make f continuous at c, thereby "removing" the discontinuity at this input value. Take a moment to identify the input values that exhibit this type of discontinuity in the figure above.

2. If $\lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x)$, we day that f has a **jump discontinuity** at c. Take a moment to identify the input values that exhibit this type of discontinuity in the figure above.

3. If either $\lim_{x \to c^-} f(x)$ or $\lim_{x \to c^+} f(x)$ attains an infinite value (∞ or $-\infty$), then we say that f has an **infinite discontinuity** at c. Take a moment to identify the input values that exhibit this type of discontinuity in the figure above.

4. The fourth type of discontinuity that can occur is a bit harder to explain. It is often called discontinuity due to **oscillation**. To see an example of this type of discontinuity, consider the graph of the function $f(x) = \sin\left(\frac{1}{x}\right)$ at $x = 0$.



Useful Facts:

- A polynomial function $f(x)$ is continuous at every real number c.
- A rational function $q(x) = \dfrac{f(x)}{g(x)}$ where f and g are polynomials is continuous at every number c for which $g(c) \neq 0$.

Theorem 8: If $f(x)$ and $g(x)$ are continuous at $x = c$, then the following are also continuous at $x = c$:

- the sum $(f + g)(x)$
- the product $(fg)(x)$
- the difference $(f - g)(x)$
- the quotient $\left(\frac{f}{g}\right)(x)$, provided $g(c) \neq 0$
- $k \cdot f(x)$ for any constant k.
- $[f(x)]^n$ for any positive integer n.
- $\sqrt[n]{f(x)}$ for any positive integer n (assuming $f(x)$ is positive on an interval containing c when n is even).

Theorem 10: If $\lim_{x\to c} g(x) = b$ and f is continuous at b, then $\lim_{x\to c} f(g(x)) = f(b) = f\left(\lim_{x\to c} g(x)\right)$

Theorem 9: If g is continuous at c and if f is continuous at $b = g(c)$, then

- $\lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right) = f(g(c))$
- the composite function $f \circ g$ is continuous at c.

The Intermediate Value Theorem: If f is continuous on a closed interval $[a, b]$ and if w is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that $f(c) = w$.

Example: