Math 261

Definite Integrals Handout

Approximating Area Using Partitions:

Given a function f on an interval $[a, b]$, we can approximate area using partitions that do not necessarily have rectangles all of the same width. A *partition* P of the interval $[a, b]$ of size n is a set of numbers $a = x_0 < x_1 < x_2 < ... < x_{n-1} < x_n = b$. $\Delta x_k = x_k - x_{k-1}$ is the width of the kth subinterval, and $\|P\|$, the *norm* of the partition P, is the width of the widest of all the subintervals in P.

The *Riemann sum* of f on $[a, b]$ for a partition P is $R_P = \sum_{k=1}^{n} f(w_k)\Delta x_k$, where w_k is some point in the kth subinterval of the partition P.

If $\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(w_k)\Delta x_k = J$ for some real number J, then we say that f is integrable on $[a, b]$, and the definite integral of f on $[a, b]$ is:

$$\int_a^b f(x)dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(w_k)\Delta x_k = J$$

Theorem 1 – Integrability of Continuous Functions: If a function f is continuous over the interval $[a, b]$ then f is integrable over $[a, b]$. Similarly, if f has at most finitely any jump discontinuities and no other discontinuities on $[a, b]$, then f is integrable over $[a, b]$.

Proof: The proof of this Theorem is beyond the scope of this course.

Properties of Definite Integrals

1. $\int_a^b c\ dx = c(b - a)$
2. $\int_a^a f(x)\ dx = 0$
3. $\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$
4. $\int_a^b cf(x)\ dx = c\int_a^b f(x)\ dx$, for any constant c
5. $\int_a^b f(x) \pm g(x)\ dx = \int_a^b f(x)\ dx \pm \int_a^b g(x)\ dx$
6. $\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$
7. If f has a maximum value M on $[a, b]$ and a minimum value m on $[a, b]$, then $m \cdot (b - a) \le \int_a^b f(x)\ dx \le M \cdot (b - a)$.
8. If f is integrable on $[a, b]$ and $f(x) \ge 0$ for every x in $[a, b]$, then $\int_a^b f(x)\ dx \ge 0$
9. If f and g are integrable on $[a, b]$ and $f(x) \ge g(x)$ for every x in $[a, b]$, then $\int_a^b f(x)\ dx \ge \int_a^b g(x)\ dx$

Definitions: Let $y = f(x)$ be a function that is non-negative and integrable on an interval $[a, b]$.
Then the **area under the curve** $y = f(x)$ **over [a,b]** is the definite integral of f from a to b: $A = \int_a^b f(x)\ dx$.

Let f be a function that is integrable on an interval $[a, b]$. Then the **average value** of f over $[a, b]$ is $av(f) = \frac{1}{b-a}\int_a^b f(x)\ dx$.

The Mean Value Theorem for Definite Integrals:

If f is continuous on $[a, b]$, then there is a number c in the open interval (a, b) such that
$f(c) = \frac{1}{b-a}\int_a^b f(x)\ dx$