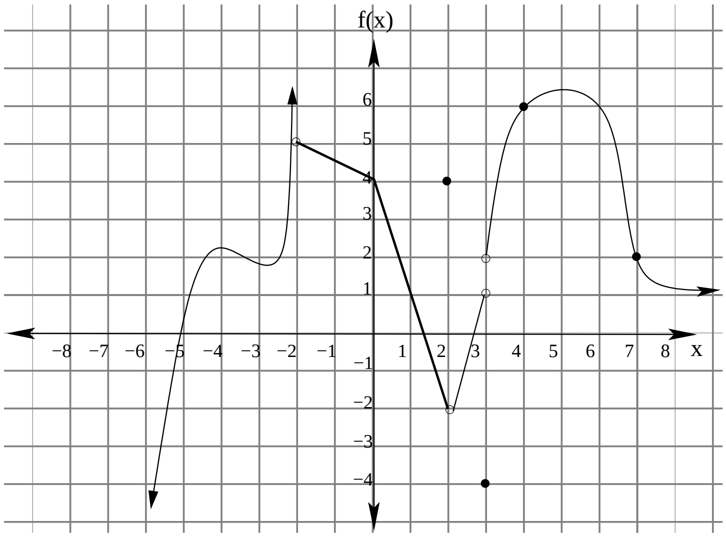
Math 261
Exam 1
09/16/2014

Name:____________________________________

Instructions: You will have 55 minutes to complete this exam. Non-graphing Calculators are allowed, but this is a closed book, closed notes exam. The credit given on each problem will be proportional to the amount of correct work shown. Correct answers without supporting work will receive little credit. Simplify answers when possible and follow directions carefully on each problem.

1. (3 points each) Use the information presented in the following graph to find each of the following:



(a) $f(2) = 4$

(b) $f(3) = -4$

(c) $\lim_{x \to -2^+} f(x) = 5$

(d) $\lim_{x \to -2^-} f(x) = \infty$

(e) $\lim_{x \to 2} f(x) = -2$

(f) $\lim_{x \to 3^+} f(x) = 2$

(g) $\lim_{x \to \infty} f(x) = 1$

(h) the average rate of change of $f(x)$ on $[4, 7]$

$$= \frac{f(7) - f(4)}{7 - 4} = \frac{2 - 6}{3} = -\frac{4}{3}$$

(i) (6 points) List all values of x at which $f(x)$ is discontinuous and classify the type of discontinuity at each of these values. You do not need to justify your answers.

There are three points of discontinuity:

$x = -2$ is an infinite type discontinuity.

$x = 2$ is a removable discontinuity.

$x = 3$ is a jump discontinuity.

2. (2 points each) Find the exact value of each of the following:

(a) $\cos\left(\frac{5\pi}{6}\right)$

$= -\cos\left(\frac{\pi}{6}\right) = -\dfrac{\sqrt{3}}{2}$

(b) $\tan\left(\frac{4\pi}{3}\right)$

$= \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$

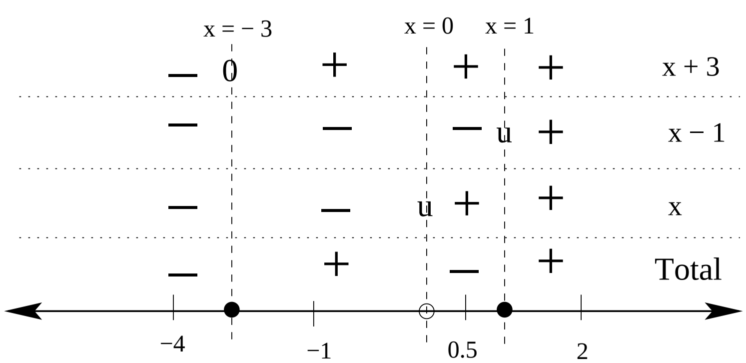
(c) $\sin^{-1}\left(-\frac{1}{2}\right)$

$= -\dfrac{\pi}{6}$

3. (6 points) Find all solutions to the inequality $\dfrac{x+3}{x(x-1)} \geq 0$

The key values are: $x = -3$, $x = 0$, and $x = 1$.

Carrying out sign testing gives the following sign chart:



Therefore, the inequality has the following solution set: $[-3, 0) \cup (1, \infty)$.

4. (3 points each) Evaluate the following limits (use the symbols ∞, $-\infty$, or DNE if appropriate). You do **not** need to justify your answers.

(a) $\lim\limits_{x\to 1} \sqrt{2}$

$= \sqrt{2}$

(b) $\lim\limits_{x\to 2^-} \dfrac{2x+1}{3x-6}$

form: $\frac{5}{0}$, signs: $\frac{(+)}{(-)}$ so $= -\infty$

(c) $\lim\limits_{x\to 3} \sqrt{5x-1}$

$= \sqrt{5(3)-1} = \sqrt{14}$

(d) $\lim\limits_{x\to 2^+} \dfrac{x-2}{1-x}$

form: $\frac{0}{-1} = 0$

(e) $\lim\limits_{x\to\infty} \dfrac{7x^2-4}{5x^2+1}$

$= \lim\limits_{x\to\infty} \dfrac{7-\frac{4}{x^2}}{5+\frac{1}{x^2}} = \dfrac{7}{5}$

(f) $\lim\limits_{x\to 0} \dfrac{\sin^2 x}{5x^2\cos(x)}$

$= \lim\limits_{x\to 0} \dfrac{1}{5} \cdot \dfrac{\sin x}{x} \cdot \dfrac{\sin x}{x} \cdot \dfrac{1}{\cos x} = \dfrac{1}{5} \cdot 1 \cdot 1 \cdot 1 = \dfrac{1}{5}$

5. (3 points each) Given that $\lim_{x\to 3} f(x) = 5$, $\lim_{x\to 3} g(x) = -2$, $\lim_{x\to 3} h(x) = 11$, and $\lim_{x\to -2} h(x) = 9$, Find the following limits.

(a) $\displaystyle\lim_{x\to 3} \frac{g(x)+3x^2}{f(x)h(x)}$

$$= \frac{-2+3\cdot 9}{5\cdot 11} = \frac{25}{55} = \frac{5}{11}$$

(b) $\displaystyle\lim_{x\to 3} \sqrt{(h\circ g)(x)}$

Recall that $(h\circ g)(x) = h(g(x))$.

Then $= \sqrt{\lim_{x\to -2} h(x)} = \sqrt{9} = 3$.

6. (6 points each) Evaluate the following limits (use the symbols ∞, $-\infty$, or DNE if appropriate). Whenever possible, use algebra to verify the value of the limit.

(a) $\displaystyle\lim_{x\to 4} \frac{x^2-16}{2-\sqrt{x}} \cdot \frac{2+\sqrt{x}}{2+\sqrt{x}}$

$$= \lim_{x\to 4} \frac{(x-4)(x+4)\left(2+\sqrt{x}\right)}{4-x}$$

$$= \lim_{x\to 4} (x+4)(-1)\left(2+\sqrt{x}\right)$$

$$= (-1)(8)(2+\sqrt{4}) = -8(4) = -32.$$

(b) $\displaystyle\lim_{x\to 2} \frac{\frac{1}{14}-\frac{1}{2x}}{x-7}$

$$= \lim_{x\to 2} \left(\frac{x}{14x} - \frac{7}{14x}\right) \cdot \frac{1}{x-7}$$

$$= \lim_{x\to 2} \left(\frac{x-7}{14x}\right) \cdot \frac{1}{x-7}$$

$$= \lim_{x\to 2} \frac{1}{14x} = \frac{1}{28}$$

7. (7 points) Find all asymptotes of the function $f(x) = \dfrac{3x^2+3x}{x^3-3x^2-4x}$

First, note that $f(x) = \dfrac{3x^2+3x}{x^3-3x^2-4x} = \dfrac{3x(x+1)}{x(x-4)(x+1)}$.

From this, we see that $f(x)$ has a removable discontinuities when $x=-1$ and $x=0$, and an infinite type discontinuity at $x=4$.

Hence $f(x)$ has a vertical asymptote at $x=4$.

Next, notice that $\displaystyle\lim_{x\to\infty} \frac{3x^2+3x}{x^3-3x^2-4x} = \lim_{x\to\infty} \frac{\frac{3}{x}+\frac{3}{x^2}}{1-\frac{3}{x}-\frac{4}{x^2}} = \frac{0}{1} = 0.$

Therefore, $f(x)$ has a horizontal asymptote: $y=0$.

8. (5 points) Let $g(t) = \begin{cases} 3t^2 - 2 & t < 2 \\ \sqrt{5t^3 + \ell} & t \geq 2 \end{cases}$. Find a value for ℓ so that $g(t)$ is continuous at $t = 2$.

Notice that $\lim_{t \to 2^-} g(t) = 3(4) - 2 = 10$. Similarly, $\lim_{t \to 2^+} g(t) = \sqrt{40 + \ell}$

Then in order for $g(t)$ to be continuous at $t = 2$, we must have $\sqrt{40 + \ell} = 10$, or $40 + \ell = 100$. Thus $\ell = 60$.

Notice that when $\ell = 60$, then $g(2) = \sqrt{5 \cdot 2^3 + 60} = \sqrt{40 + 60} = \sqrt{100} = 10$. Hence, when $\ell = 60$, then $g(t)$ is continuous at $t = 2$.

9. (5 points) State the formal definition of $\lim_{x \to a} f(x) = L$.

Let f be a function defined on an open interval containing a, except possibly at a itself, and let L be a real number. The statement $\lim_{x \to a} f(x) = L$ means that for every $\epsilon > 0$ there is a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.

10. (7 points) Use the formal definition of a limit to prove that $\lim_{x \to 2} 5 - 2x = 1$.

Let $\epsilon > 0$ and suppose that $|f(x) - L| < \epsilon$. Then $|(5 - 2x) - 1| = |4 - 2x| < \epsilon$.

But then $2|2 - x| = 2|x - 2| < \epsilon$, so $|x - 2| < \frac{\epsilon}{2}$.

Therefore, let $\delta \leq \frac{\epsilon}{2}$, and suppose $0 < |x - 2| < \delta \leq \frac{\epsilon}{2}$.

Then $2|x - 2| = 2|2 - x| < 2 \cdot \frac{\epsilon}{2} = \epsilon$.

Therefore $|4 - 2x| = |(5 - 2x) - 1| < \epsilon$, or $|f(x) - L| < \epsilon$.

Thus $\lim_{x \to 2} 5 - 2x = 1$ $\square$.