Math 261
Exam 1 - Practice Problem Solutions

1. Given the points $A : (4, -2)$ and $B : (-2, 7)$:

   (a) Find an equation for the line containing A and B
   $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{7 - (-2)}{-2 - 4} = \dfrac{9}{-6} = -\dfrac{3}{2}$.
   Using the point $(-2, 7)$ and the point-slope formula, $y - (7) = -\frac{3}{2}(x + 2)$, or $y - 7 = -\frac{3}{2}x - 3$.
   Therefore, $y = -\frac{3}{2}x + 4$

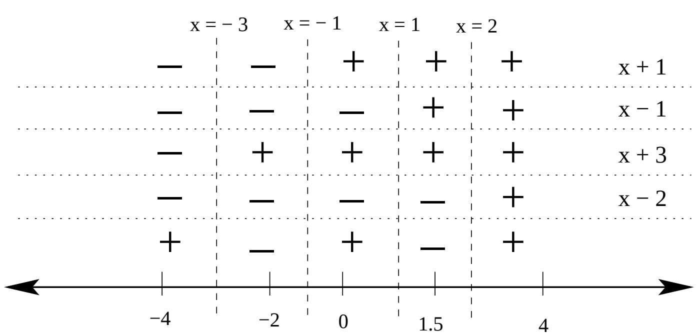
   (b) Find the line that is perpendicular to the line you found in part (a) and containing the point $(1, -1)$
   Since the slope of the previous line is $m_1 = -\frac{3}{2}$, a line that is perpendicular to the previous line has slope equal to the negative reciprocal $m_2 = -\frac{1}{m_1} = \frac{2}{3}$.
   Also, since the line passes through $(1, -1)$, the equation of the line is given by:
   $y + 1 = \frac{2}{3}(x - 1) = \frac{2}{3}x - \frac{2}{3}$
   Thus, $y = \frac{2}{3}x - \frac{5}{3}$

2. Find solutions to the inequality: $\dfrac{x^2 - 1}{x^2 + x - 6} \leq 0$.

   Factoring, we have: $\dfrac{(x + 1)(x - 1)}{(x + 3)(x - 2)} \leq 0$.

   Notice that the numerator is zero when $x = 1$ or $x = -1$, and the denominator is zero when $x = -3$ or $x = 2$.

   Therefore, using sign analysis, we have the following sign diagram:



   Thus the solution to this inequality, in interval notation, is: $(-3, -1] \cup [1, 2)$

3. Given the function $f(x) = \dfrac{1}{x - 2}$

   (a) What is the domain of f? Give your answer in interval notation.
   Notice that $f(x)$ is defined except when $x = 2$. Therefore, the domain, in interval notation, is: $(-\infty, 2) \cup (2, \infty)$

   (b) Find $f(5)$ and $f(2a + 4)$
   $f(5) = \frac{1}{5-2} = \frac{1}{3}$. Similarly, $f(2a + 4) = \frac{1}{(2a+4)-2} = \frac{1}{2a+2} = \frac{1}{2(a+1)}$

   (c) Find $\dfrac{f(a + h) - f(a)}{h}$ (be sure to simplify your answer).
   $$\frac{f(a + h) - f(a)}{h} = \frac{\frac{1}{a+h-2} - \frac{1}{a-2}}{h} = \frac{\frac{a-2}{a+h-2} - \frac{a+h-2}{a-2}}{h} = \frac{(a - 2) - (a + h - 2)}{(a + h - 2)(a - 2)} \cdot \frac{1}{h} = \frac{a - 2 - a - h + 2}{(a + h - 2)(a - 2)} \cdot \frac{1}{h} =$$
   $$\frac{-h}{(a + h - 2)(a - 2)} \cdot \frac{1}{h} = \frac{-1}{(a + h - 2)(a - 2)}$$

4. Given that $f(x) = \dfrac{1}{2x-3}$ and $g(x) = \sqrt{x^2 - 9}$

   (a) Find $f \circ g(2)$
   $f \circ g(2) = f(g(2)) = f(\sqrt{4-9}) = f(\sqrt{-5})$, which is undefined.

   (b) Find the domain of $\frac{g}{f}$? Give your answer in interval notation.

   To be in the domain of $\frac{g}{f}$, an x-value must be in the domain of both $f(x)$ and $g(x)$, and we must also have $g(x) \neq 0$.
   The domain of $f(x)$ is all x except when $2x - 3 = 0$, or $2x = 3$. Thus, the domain is $x \neq \frac{3}{2}$
   The domain of $g(x)$ is all x for which $x^2 - 9 \geq 0$, or $x^2 \geq 9$. Thus, we need $|x| \geq 3$. Hence $x \geq 3$ or $x \leq -3$.
   Finally, we need $g(x) \neq 0$, so $x^2 \neq 9$, or $x \neq 3$ and $x \neq -3$
   Combining these, the domain is: $(-\infty, -3) \cup (3, \infty)$

5. Find the exact value of each of the following:

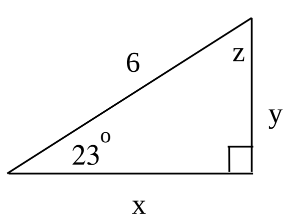
   (a) $\sin\left(\frac{7\pi}{4}\right) = -\frac{\sqrt{2}}{2}$
   (b) $\cos\left(\frac{4\pi}{3}\right) = -\frac{1}{2}$
   (c) $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$
   (d) $\cos^{-1}(-1) = \pi$
   (e) $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$

6. Find all solutions to the following equations. Give the exact answers.

   (a) $2\sin 3x = \sqrt{3}$
   $\sin 3x = \frac{\sqrt{3}}{2}$, so either $3x = \frac{\pi}{3} + 2\pi k$ or $3x = \frac{2\pi}{3} + 2\pi k$
   Hence $x = \frac{\pi}{9} + \frac{2\pi}{3}k$ or $x = \frac{2\pi}{9} + \frac{2\pi}{3}k$

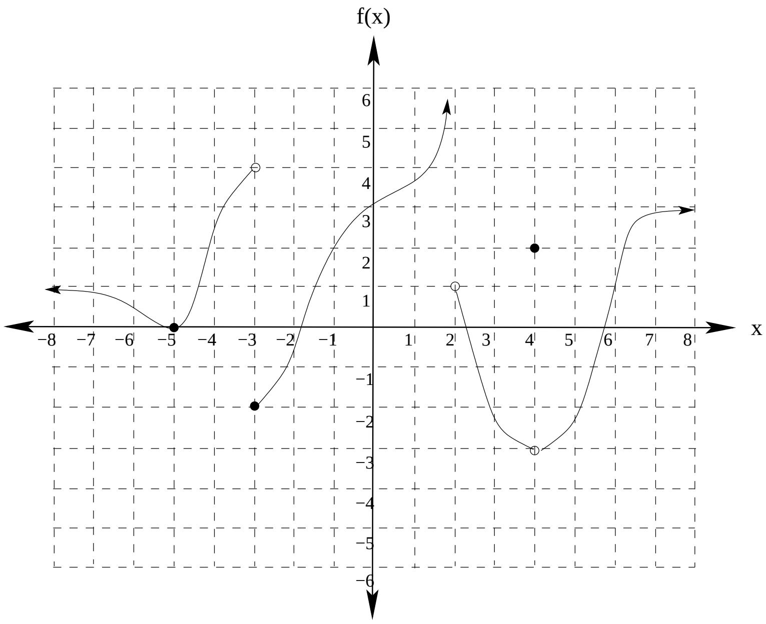
   (b) $\sin^2(x) - \sin(x) = 0$
   Factoring, $\sin x(\sin x - 1) = 0$, so $\sin x = 0$ or $\sin x = 1$
   Therefore, $x = 0 + 2\pi k$ or $x = \pi + 2\pi k$ or $x = \frac{\pi}{2} + 2\pi k$

7. Find the values of x, y and z in the triangle shown below:



First, $z = 180 - 23 - 90 = 67°$
Next, $\sin 23° = \frac{y}{6}$, so $y = 6\sin 23° \approx 2.3444$
Similarly, $\cos 23° = \frac{x}{6}$, so $x = 6\cos 23° \approx 5.5230$ (or we could use the Pythagorean Theorem to find the third side).

8. A function f is graphed below. Find the following:



(a) $f(-5)$, $f(-3)$, and $f(4)$

From the graph we see $f(-5) = 0$, $f(-3) = -2$, and $f(4) = 2$

(b) find the domain and range of f

Domain: $(-\infty, 2) \cup (2, \infty)$
Range: $(-3, \infty)$

(c) find the intervals where f is decreasing

Decreasing: $(-\infty, -5) \cup (2, 4)$

(d) find $\lim_{x \to 4} f(x)$

$\lim_{x \to 4} f(x) = -3$

(e) find $\lim_{x \to 2^-} f(x)$ and $\lim_{x \to 2^+} f(x)$

$\lim_{x \to 2^-} f(x) = \infty$ and $\lim_{x \to 2^+} f(x) = 1$

(f) find $\lim_{x \to -\infty} f(x)$ and $\lim_{x \to \infty} f(x)$

$\lim_{x \to -\infty} f(x) = 1$ and $\lim_{x \to \infty} f(x) = 3$

(g) find the points where $f(x)$ is discontinuous, and classify each point of discontinuity.

Points of discontinuity:

$x = -3$ (jump discontinuity)

$x = 2$ (infinite discontinuity)

$x = 4$ (removable discontinuity)

9. Find the following limits:

(a) $\lim_{x \to 2} \frac{3x+7}{\sqrt{5x-1}} = \frac{3(2)+7}{\sqrt{5(2)-1}} = \frac{13}{\sqrt{9}} = \frac{13}{3}$

(b) $\lim_{x \to \frac{3}{2}} \frac{2x^2+x-6}{4x^2-4x-3} = \lim_{x \to \frac{3}{2}} \frac{(2x-3)(x+2)}{(2x-3)(2x+1)} = \lim_{x \to \frac{3}{2}} \frac{x+2}{2x+1} = \frac{\frac{3}{2}+2}{(2)\frac{3}{2}+1} = \frac{\frac{7}{2}}{4} = \frac{7}{8}$

(c) $\lim_{x \to 2} \frac{x^4-16}{x^2-x-2} = \lim_{x \to 2} \frac{(x^2+4)(x^2-4)}{(x-2)(x+1)} = \lim_{x \to 2} \frac{(x^2+4)(x-2)(x+2)}{(x-2)(x+1)} = \lim_{x \to 2} \frac{(x^2+4)(x+2)}{x+1}$
$= \frac{(2^2+4)(2+2)}{2+1} = \frac{(8)(4)}{3} = \frac{32}{3}$

(d) $\lim_{x \to -2^+} \sqrt{x+2}$

Notice that $\sqrt{x+2}$ is defined for $x \geq -2$. Therefore, $\lim_{x \to -2^+} \sqrt{x+2} = \sqrt{-2+2} = 0$

(e) $\lim_{x \to 3^+} \frac{4}{\sqrt{x-3}}$

Notice that for $x > 3$, $\sqrt{x-3} > 0$. Therefore, $\lim_{x \to 3^+} \frac{4}{\sqrt{x-3}} = \infty$

(f) $\lim_{x \to \infty} \frac{(3x-5)(2x-3)}{(2x+1)(3x-2)}$

$\lim_{x \to \infty} \frac{(3x-5)(2x-3)}{(2x+1)(3x-2)} = \lim_{x \to \infty} \frac{6x^2-19x+15}{6x^2-x-2} = \lim_{x \to \infty} \frac{x^2(6-\frac{19}{x}+\frac{15}{x^2})}{x^2(6-\frac{1}{x}-\frac{2}{x^2})} = \lim_{x \to \infty} \frac{(6-\frac{19}{x}+\frac{15}{x^2})}{(6-\frac{1}{x}-\frac{2}{x^2})} = \frac{6}{6} = 1$

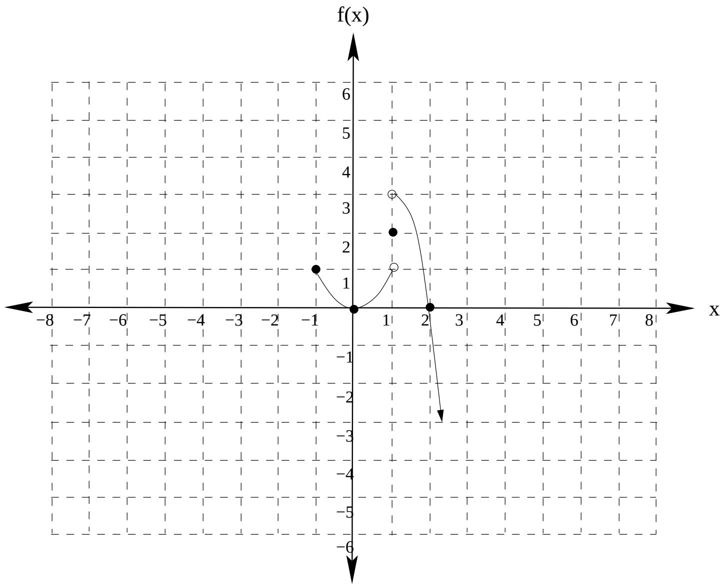
(g) $\lim_{x \to \infty} \frac{(3x-5)(2x-3)}{(2x+1)}$

$\lim_{x \to \infty} \frac{(3x-5)(2x-3)}{(2x+1)} = \lim_{x \to \infty} \frac{6x^2-19x+15}{(2x+1)} = \lim_{x \to \infty} \frac{6x-19+\frac{15}{x}}{(2+\frac{1}{x})} = \lim_{x \to \infty} \frac{6x-19}{(2)} = \lim_{x \to \infty} 3x - \frac{19}{2} = \infty$

10. Given the function

$$f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ 2 & \text{if } x = 1 \\ 4 - x^2 & \text{if } x > 1 \end{cases}$$

(a) Graph $f(x)$.



(b) Find $\lim_{x\to 1^-} f(x)$, $\lim_{x\to 1^+} f(x)$, and $\lim_{x\to 1} f(x)$

$\lim_{x\to 1^-} f(x) = 1$

$\lim_{x\to 1^+} f(x) = 3$

$\lim_{x\to 1} f(x)$ does not exist

(c) Is $f(x)$ continuous at $x = 1$? Justify your answer.

No. Since $\lim_{x\to 1} f(x)$ does not exist, $f(x)$ is **not** continuous at $x = 1$.

11. Given that $f(x) = x^3 + 5$, $\lim_{x\to 2} f(x) = 13$, and $\epsilon = .01$, find the largest δ such that if $0 < |x - 2| < \delta$, then $|f(x) - 13| < \epsilon$.

If $|f(x) - 13| < \epsilon$, then $|x^3 + 5 - 13| < \epsilon$, or $|x^3 - 8| < .01$

That is, $-.01 < x^3 - 8 < .01$, or $7.99 < x^3 < 8.01$. Thus $\sqrt[3]{7.99} < x < \sqrt[3]{8.01}$

Notice that $2 - \sqrt[3]{7.99} \approx -.000833681$ and $\sqrt[3]{8.01} - 2 \approx .000832986$

Then the largest δ that works is $\delta = \sqrt[3]{8.01} - 2$

12. Use the formal definition of a limit to prove that $\lim_{x \to 6} 5x - 21 = 9$.

Let $\epsilon > 0$ be givenm and suppose that $|f(x) - 9| < \epsilon$. Then $|5x - 21 - 9| = |5x - 30| < \epsilon$.
But then $5|x - 6| < \epsilon$, so $|x - 6| < \frac{\epsilon}{5}$.

Therefore, let $\delta \leq \frac{\epsilon}{5}$, and suppose $|x - 6| < \delta$.

Then $5|x - 6| < 5\delta \leq \epsilon$.
Therefore $|5x - 30| = |5x - 21 - 9| < \epsilon$, or $|f(x) - 9| < \epsilon$.

Thus $\lim_{x \to 6} 5x - 9 = 21$

13. Let $f(x) = \dfrac{x^2 - x - 2}{x^2 - 2x}$.

(a) Find the values of x at which f is discontinuous.
Factoring, $f(x) = \dfrac{x^2 - x - 2}{x^2 - 2x} = \dfrac{(x-2)(x+1)}{x(x-2)}$
Therefore, we can see that $f(x)$ is discontinuous at $x = 0$ and $x - 2$

(b) Find all vertical and horizontal asymptotes of f.
Since we can cancel the two $(x - 2)$ terms, there is **not** a vertical asymptote when $x = 2$
The only vertical asymptote is at $x = 0$.
To find the horizontal asymptote, we compute $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \dfrac{x^2 - x - 2}{x^2 - 2x} = \lim_{x \to \infty} \dfrac{1 - \frac{1}{x} - \frac{2}{x^2}}{1 - \frac{2}{x}} = 1$.
Thus, $y = 1$ is the horizontal asymptote of $f(x)$.

14. Find the x values at which $f(x) = \dfrac{\sqrt{9 - x^2}}{x^4 - 16}$ is continuous.

First, notice that that $x^4 - 16 = (x^2 + 4)(x^2 - 4) = (x^2 + 4)(x - 2)(x + 2)$, so $f(x)$ is undefined when $x = 2$ and $x = -2$.
Also, for $f(x)$ to be defined, we must have $9 - x^2 \geq 0$, or $x^2 \leq 9$. Thus $-3 \geq x \geq 3$.
Therefore, $f(x)$ is continuous on the intervals: $[-3, -2) \cup (-2, 2) \cup (2, 3]$

15. Use the Intermediate Value Theorem to show $x^5 - 3x^4 - 2x^3 - x + 1 = 0$ has a solution between 0 and 1.

Let $f(x) = x^5 - 3x^4 - 2x^3 - x + 1$. Notice that f is continuous since it is a polynomial. Also, $f(0) = 1$ and $f(1) = -4$.
Thus, by the IVT, for every $-4 < w < 1$, there is a c satisfying $0 \leq c \leq 1$ with $f(c) = w$. In particular, $f(c) = 0$ for some c between zero and 1.