Math 261
Exam 2 - Practice Problems

1. Find the derivative $y' = \frac{dy}{dx}$ for each of the following:

   (a) $y = \pi^2 x + \pi x^2$
   $y' = \pi^2 + 2\pi x$

   (b) $y = \cot x$
   $y' = -\csc^2 x$

   (c) $y = \sqrt{x}\sec(x^2)$
   $y' = \frac{1}{2}x^{-\frac{1}{2}}\sec(x^2) + x^{\frac{1}{2}}\sec(x^2)\tan(x^2)\cdot 2x$
   $$y' = \frac{1}{2\sqrt{x}}\sec(x^2) + 2x\sqrt{x}\sec(x^2)\tan(x^2)$$

   (d) $y = 2\tan^3(2x^3)$
   $y' = 6\tan^2(2x^3)\cdot\sec^2(2x^3)\cdot 6x^2 = 36x^2\tan^2(2x^3)\sec^2(2x^3)$

   (e) $$y = \frac{x^2 - 7\cos(3x)}{x + \sin(3-2x)}$$
   $$y' = \frac{(2x + 21\sin(3x))(x + \sin(3-2x)) - (x^2 - 7\cos(3x))(1 - 2\cos(3-2x))}{(x + \sin(3-2x))^2}$$
   Note: I won't make you take time to simplify problems like this one on the exam.

   (f) $x^2 y + 3xy - 5y^2 = 7$
   Differentiating with respect to x: $2xy + x^2 y' + 3y + 3xy' - 10yy' = 0$
   Then $(x^2 + 3x - 10y)y' = -2xy - 3y$
   Thus $$y' = \frac{-2xy - 3y}{x^2 + 3x - 10y}$$

   (g) $\cos^2(xy) = 1$
   Differentiating with respect to x: $y' = 2\cos(xy)\cdot(-\sin(xy))\cdot(y + xy') = 0$
   Then $-2y\cos(xy)\sin(xy)) - [2x\cos(xy)\sin(xy))]y' = 0$,
   or $-y'[2x\cos(xy)\sin(xy))] = 2y\cos(xy)\sin(xy)$
   Thus $$y' = \frac{2y\cos(xy)\sin(xy)}{-2x\cos(xy)\sin(xy)} = -\frac{y}{x}$$

2. Use the formal limit definition of the derivative to find the derivative of the following:

   (a) $f(x) = x^2 - 3x$
   $$f'(x) = \lim_{h\to 0}\frac{(x+h)^2 - 3(x+h) - x^2 + 3x}{h} = \lim_{h\to 0}\frac{x^2 + 2xh + h^2 - 3x - 3h - x^2 + 3x}{h}$$
   $$= \lim_{h\to 0}\frac{2xh + h^2 - 3h}{h} = \lim_{h\to 0} 2x + h - 3 = 2x - 3$$

(b) $f(x) = \dfrac{2}{x-3}$

$$f'(x) = \lim_{h\to 0} \frac{\frac{2}{x+h-3} - \frac{2}{x-3}}{h} = \lim_{h\to 0} \frac{\frac{2(x-3)-2(x+h-3)}{(x+h-3)(x-3)}}{h}$$

$$= \lim_{h\to 0} \frac{2x-6-2x-2h+6}{(x+h-3)(x-3)} \frac{1}{h}$$

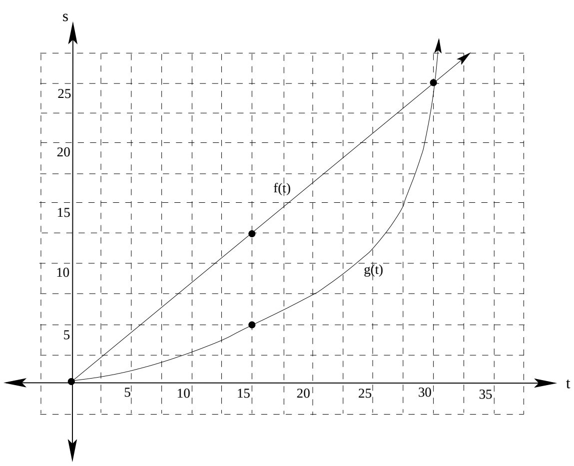
$$= \lim_{h\to 0} \frac{-2h}{(x+h-3)(x-3)} \frac{1}{h} = \lim_{h\to 0} \frac{-2}{(x+h-3)(x-3)} = \frac{-2}{(x-3)^2}$$

(c) $f(x) = \sqrt{x-2}$

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h-2} - \sqrt{x-2}}{h} \cdot \frac{\sqrt{x+h-2} + \sqrt{x-2}}{\sqrt{x+h-2} + \sqrt{x-2}}$$

$$= \lim_{h\to 0} \frac{x+h-2-x+2}{h(\sqrt{x+h-2} + \sqrt{x-2})} = \lim_{h\to 0} \frac{h}{h(\sqrt{x+h-2} + \sqrt{x-2})} = \frac{1}{2\sqrt{x-2}}$$

3. The position of two cars, car A and car B, both starting side by side on a straight road, is given by $f(t)$ and $g(t)$, where $f(t)$ is the distance traveled car A in feet, and $g(t)$ is the distance traveled car B in feet, and t is in minutes (see the graph below):



(a) How fast is car A going at time $t = 15$?

To find the speed of car A at time $t = 15$, we need to find the slope of the tangent line to $f(t)$ when $t = 15$. From the graph, $m = \frac{12.5}{15} = \frac{5}{6}$ feet per minute.

(b) Find the average rate of change of car B on the time interval $[0, 15]$.

The average rate of change of car B on the time interval $[0, 15]$ is given by the slope of the secant line to $g(t)$, which is given by $v_{av} = \frac{5}{15} = \frac{1}{3}$ feet per minute.

(c) Which car is traveling faster at time $t = 15$?

We are comparing the instantaneous velocities of the two cars when $t = 15$. From the graph, we see that the tangent line to $f(t)$ is steeper than the tangent line to $g(t)$ when $t = 15$, so car A is going faster at that time.

(d) Which car is traveling faster at time $t = 30$?

We are comparing the instantaneous velocities of the two cars when $t = 30$. From the graph, we see that the tangent line to $g(t)$ is steeper than the tangent line to $f(t)$ when $t = 30$, so car B is going faster at that time.

(e) What can you say about the relative positions of the two cars at time $t = 30$?

Since the cars have driven the same distance, they are still side by side.

4. Use the quotient rule to derive the formula for the derivative of $\tan(x)$.

   Notice that $f(x) = \tan x = \dfrac{\sin x}{\cos x}$

   Then, using the quotient rule:

   $f'(x) = \dfrac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x} = \dfrac{\cos^2 x + \sin^2 x}{\cos^2 x} = \dfrac{1}{\cos^2 x} = \sec^2(x).$

   Thus $\frac{d}{dx}(\tan x) = \sec^2 x$

5. Use the product rule to prove that $D_x[f(x)g(x)h(x)] = f'(x)g(x)h(x)+f(x)g'(x)h(x)+f(x)g(x)h'(x)$

   We'll use the product rule twice:

   $D_x[(f(x)g(x))\, h(x)] = D_x[f(x)g(x)]h(x) + (f(x)g(x)) \cdot h'(x)$

   $= [f'(x)g(x) + f(x)g'(x)]h(x) + f(x)g(x)h'(x) = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$

   □

6. If $f(x) = \sqrt{3x-5}$, find the intervals where $f(x)$ is continuous, and find the intervals where $f(x)$ is differentiable.

   Recall that $f(x)$ is the square root of a polynomial, so it is continuous wherever it is defined. That is, whenever $3x - 5 \geq 0$, or when $x \geq \frac{5}{3}$.

   Thus $f(x)$ is continuous on $[\frac{5}{3}, \infty)$

   Next, $f'(x) = \dfrac{1}{2}(3x-5)^{-\frac{1}{2}}(3) = \dfrac{3}{2\sqrt{3x-5}}$

   Then $f'(x)$ is defined when $3x - 5 > 0$, or when $x > \frac{5}{3}$, so $f(x)$ is differentiable on $(\frac{5}{3}, \infty)$

7. If $f(x) = 3x^4 - 5x^2 + 7x - 12$, use differentials to approximate $f(1.1)$

   Let $x = 1$ and $\Delta x = .1$. Notice that $f'(x) = 12x^3 - 10x + 7$, so $f'(1) = 12 - 10 + 7 = 9$, and $f(1) = 3 - 5 + 7 - 12 = 10 - 17 = -7$

   Then $f(1.1) \approx f(1) + f'(1)\Delta x = -7 + 9(.1) = -7 + .9 = -6.1$

8. Use differentials to approximate $\sqrt{1.2}$. How good is your approximation?

   Let $f(x) = \sqrt{x}$. Then $f'(x) = \frac{1}{2\sqrt{x}}$.

   Let $x = 1$ and $\Delta x = .2$. Then, using the linear approximation formula:

   $f(1.2) \approx f(1) + f'(1)\Delta x = 1 + \frac{1}{2} \cdot (.2) = 1.1$

   Notice that using a calculator, $\sqrt{1.2} \approx 1.095445$, so if we believe our calculator, our approximation using the tangent line to f when $x = 1$ is good to within about .004556.

9. Use differentials to estimate $\sqrt[3]{9}$. How good is your approximation?

   Let $f(x) = x^{\frac{1}{3}}$, $x = 8$, and $\Delta x = 1$.

   Then $f'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3x^{\frac{2}{3}}}$.

   Therefore, $f(8) = \sqrt[3]{8} = 2$, and $f'(8) = \frac{1}{3 \cdot 8^{\frac{2}{3}}} = \frac{1}{3(4)} = \frac{1}{12}$.

   Thus $f(9) \approx f(8) + f'(8)\Delta x = 2 + \frac{1}{12} = \frac{25}{12} \approx 2.08333$

   Notice that $\sqrt[3]{9} \approx 2.08008$, so our approximation in within 33 ten-thousandths.

10. Suppose helium is being pumped into a spherical balloon at a rate of 4 cubic feet per minute. Find the rate at which the radius is changing when the radius is 2 feet.

    Recall that the volume of a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$. In the situation described, both V and r are functions of time t in minutes.

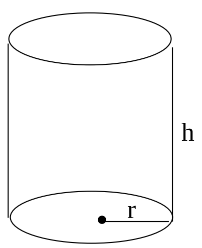
    Then, differentiating implicitly, $\dfrac{dV}{dt} = 4\pi r^2 \dfrac{dr}{dt}$.

    We also know that $\dfrac{dV}{dt} = 4\dfrac{ft^3}{min}$ and $r = 2$ feet.

    Thus $4 = 4\pi(4)\dfrac{dr}{dt}$, so $\dfrac{dr}{dt} = \dfrac{4}{16\pi} = \dfrac{1}{4\pi}\dfrac{ft}{min}$.

11. Dr. Von Klausen has just invented a shrink ray and decides to try it out on a test object: a cylinder whose height is twice its radius. The shrink ray has been calibrated so that the proportions of the cylinder remain the same throughout the test. A few seconds into the test, the radius of the cylinder is decreasing at 2 inches per second, and the height is 4 inches. At what rate is the volume of the cylinder changing at that time? (Be sure to include units in your answer)

    Recall that the volume of a cylinder is given by: $V = \pi r^2 h$

    Here, $h = 2r$ and $h = 4$, so $r = 2$. Also, $\frac{dr}{dt} = -2$ inches per second.

    Substituting $h = 2r$ into the main volume equation, we get $v = 2\pi r^3$.

    Differentiating implicitly: $\frac{dV}{dt} = 6\pi r^2 \frac{dr}{dt} = 6\pi(2^2)(-2) = -48\pi$ cubic inches per second.

12. Find the equation of the tangent line to the graph of $f(x) = \tan(4x)$ when $x = \frac{3\pi}{16}$

    $f'(x) = 4\sec^2(4x) = \dfrac{4}{\cos^2(4x)}$. Therefore $f'(\frac{3\pi}{16}) = \dfrac{4}{\cos^2(\frac{12\pi}{16})} = \dfrac{4}{\left(\frac{-\sqrt{2}}{2}\right)^2} = \dfrac{4}{\frac{1}{2}} = 8,$

    and $f(\frac{3\pi}{16}) = \tan(\frac{3\pi}{4}) = -1$.

    Then the tangent line to $f(x)$ when $x = \frac{3\pi}{16}$ has slope 8 and goes through the point $\left(\frac{3\pi}{16}, -1\right)$

    Hence the tangent line has equation $y + 1 = 8\left(x - \frac{3\pi}{16}\right)$ so $y = 8x - \frac{3\pi}{2} - 1$

13. Find the equation of the tangent line to the graph of $y = \sec(2x)$ when $x = \frac{\pi}{6}$.

    $$y' = 2\sec(2x)\tan(2x) = \frac{2\sin(2x)}{\cos^2(2x)}$$

    $$m = \frac{2\sin\left(\frac{\pi}{3}\right)}{\cos^2\left(\frac{\pi}{3}\right)} = \frac{2\left(\frac{\sqrt{3}}{2}\right)}{\left(\frac{1}{2}\right)^2} = 4\sqrt{3}$$

    $$y = \sec\frac{\pi}{3} = \frac{1}{\cos(\frac{\pi}{3})} = \frac{1}{\frac{1}{2}} = 2$$

    Therefore, the equation of the tangent line is given by: $y - 2 = 4\sqrt{3}\left(x - \frac{\pi}{6}\right)$,

    or $y = 4\sqrt{3}x - \dfrac{2\sqrt{3}\pi + 6}{3}$

14. Find the points on the graph of $y = 2x^3 + 3x^2 - 72x + 5$ at which the tangent line is horizontal.

Let $y = f(x)$. Then $f'(x) = 6x^2 + 6x - 72 = 6(x^2 + x - 12)$, so the points at which the tangent line is horizontal occur when $x^2 + x - 12 = 0$, or when $(x+4)(x-3) = 0$, that is, when $x = -4$, and $x - 3$.

Notice that $f(-4) = 2(-4)^3 + 3(-4)^2 - 72(-4) + 5 = 213$, and $f(3) = 2(3)^3 + 3(3)^2 - 72(3) + 5 = -130$

Hence the points on the graph of $y = f(x)$ with horizontal tangent lines are: $(-4, 213)$ and $(3, -130)$.

15. Find the equation of the tangent line to the graph of the relation $x^2y + 3y^2 = 3x - 7$ at the point $(2, -1)$

Differentiating implicitly: $2xy + x^2y' + 6yy' = 3$, so $y'(x^2 + 6y) = 3 - 2xy)$

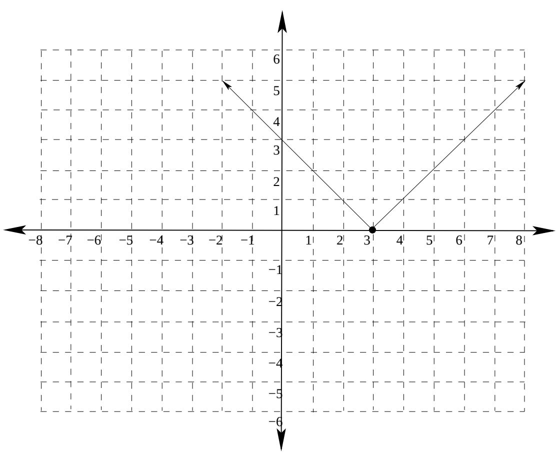
Thus $y' = \dfrac{3 - 2xy}{x^2 + 6y}$. Evaluating when $x = 2$ and $y = -1$,

$m = \dfrac{3 - 2(2)(-1)}{2^2 + 6(-1)} = \dfrac{3+4}{-2} = -\dfrac{7}{2}$.

Then the equation for the tangent line is given by: $y + 1 = -\frac{7}{2}(x - 2)$, or $y = -\frac{7}{2}x + 6$.

16. Draw the graph of a function $f(x)$ that is continuous when $x = 3$, but is not differentiable when $x = 3$.

There are many possible examples. One possibility is:



17. Find $g'(2)$ if $h(x) = f(g(x))$, $f(3) = -2$, $g(2) = 3$, $f'(3) = 5$, and $h'(2) = -30$.

Using the Chain Rule, $h'(x) = f'(g(x))g'(x)$, so $h'(2) = f'(g(2))g'(2) = f'(3)g'(2)$.

Therefore, $-30 = 5g'(2)$, so $-6 = g'(2)$.

18. Given that $f(2) = -3$, $g(2) = 2$, $f'(2) = \frac{1}{2}$, $g'(2) = -5$, and $h(x) = f(g(x))$.

Find the following:

(a) $(f - g)'(2)$

$= f'(2) - g'(2) = \dfrac{1}{2} - (-5)$

$= \dfrac{1}{2} + 5 = \dfrac{11}{2} = 5.5$

(b) $(fg)'(2)$

$= f'(2)g(2) + f(2)g'(2) = \left(\dfrac{1}{2}\right)(2) + (-3)(-5)$

$= 1 + 15 = 16$

(c) $\left(\frac{f}{g}\right)'(2)$

$= \dfrac{f'(2)g(2) - f(2)g'(2)}{[g(2)]^2}$

$= \dfrac{\left(\frac{1}{2}\right)(2) - (-3)(-5)}{2^2} = \dfrac{1 - 15}{4} = -\dfrac{7}{2}$

(d) $h'(2)$

$= f'(g(2))g'(2) = f'(2)g'(2)$

$= \left(\dfrac{1}{2}\right)(-5) = -2.5$

19. Find $f^{(8)}(x)$ if $f(x) = \sin(2x)$

    Notice that $f'(x) = 2\cos(2x)$

    Continuing in this fashion, $f^{(8)}(x) = 2^8 \sin(2x) = 256\sin(2x)$.

20. Find $f^{(13)}(x)$ if $f(x) = x^{12} + 7x^5 - 3x^3 - 1$

    Since the highest exponent is 12, and differentiation using the power rule lowers the exponent of each term by one, then $f^{(13)}(x) = 0$.

21. Use linearization to find a good approximation of $\sqrt[3]{10}$.

    There are several ways that we could accomplish this. Here is one possibility:

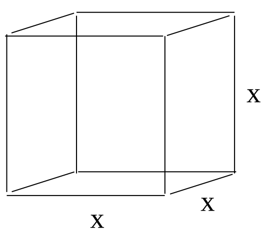
    Let $f(x) = \sqrt[3]{x+8}$ and let $a = 0$. Then $f'(x) = \frac{1}{3}(x+8)^{-\frac{2}{3}}$, so $f'(0) = \frac{1}{3}(8)^{-\frac{2}{3}} = \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$. Also note that $f(0) = \sqrt[3]{8} = 2$.

    Therefore, $L(x) = 2 + \frac{1}{12}x$. If we wish to approximate $\sqrt[3]{10}$, we must set $x = 2$.

    $L(2) = 2 + \frac{1}{12}(x) = 2 + \frac{1}{6} = \frac{13}{6}$.

    Note that $\frac{13}{6} \approx 2.16667$, while $\sqrt[3]{10} \approx 2.15443$, so we appear to be getting a decent approximation.

22. A company manufactures wooden cubes. Each side of the finished cubes are 5 inches long, with a maximum error of $\pm .2$ inches per side. Use differentials to estimate the maximum error in the volume of the cube. Then, compare your estimate with the error in volume of a cube with largest possible volume manufactured within the given error tolerances.

    The volume of a cube is given by $V = x^3$. Then $dV = 3x^2\Delta x$ can be used to approximate the error in volume. Here, $x = 5$ inches, and $\Delta x = \pm .2$ inches.

    Hence $\Delta V \approx dV = 3x^2\Delta x = 3(5^2)(\pm .2) = \pm 15$ cubic inches.

    A perfectly constructed cube would have a volume $v = 5^3 = 125$ cubic inches.

    Then, according to our estimate using differentials, $110 \leq V \leq 140$ is the error range for the volume of the manufactured cubes.

    In reality, the biggest possible cube would have sides all of length 5.2 inches, or a volume of $(5.2)^3 = 140.608$ cubic inches. So our estimate for the maximum error is pretty close to the actual maximum error.