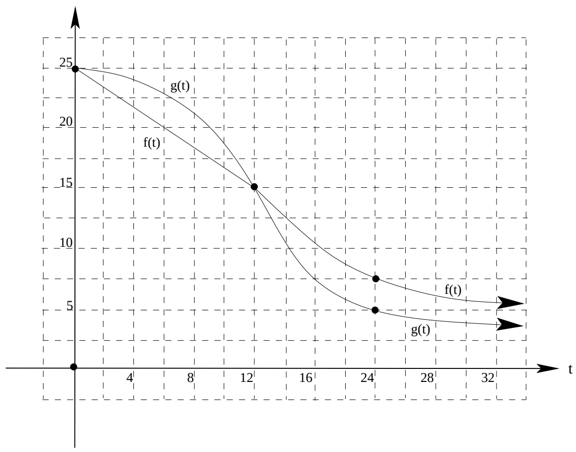
Math 261
Exam 2 **Name:**________________________________
10/07/2014

Instructions: You will have 50 minutes to complete this exam. Calculators are allowed, but no other references. The credit given will be proportional to the amount of correct work shown. Correct answers without supporting work will receive little credit. Simplify answers when possible and follow directions carefully on each problem.

1. A researcher wants to compare the effects of two antibiotics. She takes two identical bacterial cultures and treats one with antibiotic A and the other with antibiotic B. Let $f(t)$ measure the population of culture A in millions of cells and $g(t)$ the population of culture B in millions of cells, where t is measured in hours (see the graph below):



(a) (4 points) Approximately how fast is the population of culture A decreasing 6 hours after it was treated with antibiotic?

Basically, we need to find the slope of the tangent line to $f(t)$ when $t = 6$. Since this part of the graph is a line segment, we merely find the slope of the line segment using any pair of points on the segment. For example, if we take $P(0, 25)$ and $Q(12, 15)$, we see $m = \frac{25-15}{0-12} = -\frac{10}{12} = -\frac{5}{6}$ million cells per hour.

(b) (4 points) What was the average rate of change of the population of culture B from the 12th through the 24th hour after being treated?

Notice that here, since we are looking for an average rate of change, we mus compute the slope of the secant line between the two endpoints of the interval in question, $(12, 15)$, and $(24, 5)$.
That is, $m = \frac{15-5}{12-24} = \frac{10}{-12} = -\frac{5}{6}$ million cells per hour.

(c) (4 points) Which culture's population is decreasing fastest 2 hours after the antibiotic treatments begin?

By comparing the tangent lines to $f(t)$ and $g(t)$ when $t = 2$, we see that $f(t)$ is steeper (in the negative direction), so Culture A is decreasing faster after 2 hours.

(d) (4 points) Which culture's population is decreasing fastest 24 hours after the antibiotic treatments begin?

By comparing the tangent lines to $f(t)$ and $g(t)$ when $t = 24$, we see that $f(t)$ is once again steeper (in the negative direction), so Culture A is decreasing faster after 4 hours. [Since the slopes are fairly close, I gave credit to people who said the rates were the same, but a careful look shows that $f(t)$ is slightly steeper]

2. (10 points) Use the definition of the derivative to find $f'(x)$ when $f(x) = \sqrt{3x+1}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{3(x+h)+1} - \sqrt{3x+1}}{h} \cdot \frac{\sqrt{3(x+h)+1} + \sqrt{3x+1}}{\sqrt{3(x+h)+1} + \sqrt{3x+1}}$$

$$= \lim_{h\to 0} \frac{3(x+h)+1-(3x+1)}{h(\sqrt{3(x+h)+1} + \sqrt{3x+1})} = \lim_{h\to 0} \frac{3x+3h+1-3x-1}{h(\sqrt{3(x+h)+1} + \sqrt{3x+1})} = \lim_{h\to 0} \frac{3h}{h(\sqrt{3(x+h)+1} + \sqrt{3x+1})}$$

$$= \lim_{h\to 0} \frac{3}{(\sqrt{3(x+h)+1} + \sqrt{3x+1})} = \frac{3}{\sqrt{3(x+(0))+1} + \sqrt{3x+1}} = \frac{3}{2\sqrt{3x+1}}$$

3. (4 points each)

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	7	0	12	4
1	0	3	-2	-1
2	1	4	5	-3

Given that $h(x) = g(f(x))$, find the following based on the table above:

(a) $(fg)'(2)$

Since $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$

Then $(fg)'(2) = f'(2)g(2) + f(2)g'(2)$

$= (4)(5) + (1)(-3) = 20 - 3 = 17$

(b) $\left(\frac{f}{g}\right)'(2)$

Since $(f/g)'(x) = \frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$

Then $(f/g)'(2) = \frac{f'(2)g(2)-f(2)g'(2)}{[g(2)]^2}$

$\frac{(4)(5)-(1)(-1)}{[5]^2} = \frac{20-(-3)}{25} = \frac{23}{25}$

(c) $h(2)$

Since $h(x) = g(f(x))$

$h(2) = g(f(2)) = g(1) = -2$

(d) $h'(2)$

Since $h(x) = g(f(x))$, $h'(x) = g'(f(x))f'(x)$

$h'(2) = g'(f(2))f'(2) = g'(1)f'(2) = (-1)(4) = -4$

4. (8 points) Find the equation of the tangent line to the graph of $y = \tan^2(x)$ when $x = \frac{\pi}{4}$.

First, notice that when $x = \frac{\pi}{4}$, $y = \tan^2(\frac{\pi}{4}) = (1)^2 = 1$

Next, $y' = 2\tan(x)\sec^2(x)$, so when $x = \frac{\pi}{4}$, $y' = 2\tan(\frac{\pi}{4})\sec^2(\frac{\pi}{4}) = \frac{2\tan(\frac{\pi}{4})}{\cos^2(\frac{\pi}{4})} = \frac{2(1)}{[\frac{\sqrt{2}}{2}]^2} = \frac{2}{\frac{1}{2}} = 4$.

Then, applying point/slope with $P(\frac{\pi}{4}, 1)$ and $m = 4$, the tangent line to this graph when $x = \frac{\pi}{4}$ is given by:

$y - 1 = 4(x - \frac{\pi}{4})$, or $y = 4x - \pi + 1$.

5. Given the relation $y^3 - 3x^2y^2 = 0$:

(a) (5 points) Find all the points on this curve with x-coordinate 1.

Setting $x = 1$ in the equation given above, we have $y^3 - 3(1)^2y^2 = 0$, or $y^3 - 3y^2 = 0$

Then $y^2(y-3) = 0$, so either $y = 0$, or $y = 3$.

Thus $(1, 0)$ and $(1, 3)$ are the points on this curve with x-coordinate 1.

(b) (6 points) Find y' by implicitly differentiating $y^3 - 3x^2y^2 = 0$

Differentiating, we get $3y^2y' - (6xy^2 + 6x^2yy' = 0)$, or $3y^2y' - 6xy^2 - 6x^2yy' = 0)$.

Then $3y^2y' - 6x^2yy' = 6xy^2$, or $y'(3y^2 - 6x^2y) = 6xy^2$.

Thus $y' = \frac{6xy^2}{3y^2-6x^2y}$ (Note that we can simplify this to obtain $\frac{2xy}{y-2x^2}$).

(c) (5 points) Find the equation for the tangent line to the curve at the point(s) you found in part (a)

At $(1, 3)$, $y' = \frac{6(1)(3)^2}{3(3)^2-6(1)^2(3)} = \frac{54}{27-18} = \frac{54}{9} = 6$

Then the tangent line has equation $y - 3 = 6(x - 1)$, or $y = 6x - 3$

I didn't take points off if you missed this, but if we just plug in $(1, 0)$,

$y' = \frac{6(1)(0)^2}{3(0)^2-6(1)^2(0)} = \frac{0}{0}$, which is indeterminate.

If we reduce the equation of y', yielding, $y' = \frac{2xy}{y-2x^2}$, then $y' = \frac{0}{-2} = 0$, and we see that the tangent line is the horizontal line $y = 0$.

6. (8 points) Use the quotient rule to derive the standard formula for the derivative of $\csc(x)$. [Hint: $\csc x = \frac{1}{\sin x}$]

Using the quotient rule, $\frac{d}{dx}\left(\frac{1}{\sin x}\right) = \frac{0-(1)\cos x}{\sin^2 x} = \frac{-\cos x}{\sin^2 x} = -\frac{1}{\sin x}\frac{\cos x}{\sin x} = -\csc x \cot x$

7. (8 points) Use an appropriate linearization to find an approximation for $\sqrt{17}$. How good is your approximation?

Let $f(x) = \sqrt{x}$, take $a = 16$, and $\Delta x = 1$

Then $f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$.

Therefore, $f(17) \approx f(16) + f'(16)\Delta x = \sqrt{16} + \frac{1}{2\sqrt{16}} = 4 + \frac{1}{8} = 4.125$

Using a calculator, $\sqrt{17} \approx 4.123105626$, so our approximation is to within about 2 thousandths.

8. (8 points each) Find the derivative of each of the following functions. You **do not** need to simplify your answers.

   (a) $f(x) = \dfrac{\tan^2(x)}{\sqrt{x^3 - 2x + 1}}$

   $f'(x) = \frac{2\tan(x)\sec^2(x)\sqrt{x^2+1} - \tan^2(x)\frac{1}{2}(x^2+1)^{-\frac{1}{2}}(2x)}{\left(\sqrt{x^2+1}\right)^2}$

   $= \frac{2\tan(x)\sec^2(x)\sqrt{x^2+1} - x\tan^2(x)(x^2+1)^{-\frac{1}{2}}}{x^2+1}$

   (b) $g(x) = x^5 \sec^3(2x)$

   $g'(x) = 4x^3\sec^2(5x) + x^4 2(\sec(5x))(\sec(5x)\tan(5x))(5) = 4x^3\sec^2(5x) + 10x^4\sec^2(5x)\tan(5x)$

9. (8 points) In calm waters, oil spilling from the ruptured hull of a tanker spreads in all directions. If the area polluted by the spill is a circle and its radius is increasing at a rate of 2 ft/sec, how fast is the area increasing when the radius of the spill is 40 feet? (Be sure to include units in your answer) [Recall, the area of a circle is given by the equation $A = \pi r^2$]

   Notice that the area of the oil spill is given by $A = \pi r^2$, so, differentiating implicitly:

   $\frac{dA}{dt} = 2\pi r \frac{dr}{dt} = 2\pi(40ft)(2ft/sec.) = 160\pi ft^2 \sec \approx 502.655 ft^2/sec.$