Math 261
Exam 4 - Version 2 **Name:**________________________________
12/05/2014

Instructions: You will have 55 minutes to complete this exam. Calculators are allowed, but this is a closed book, closed notes exam. The credit given on each problem will be proportional to the amount of correct work shown. Correct answers without supporting work will receive little credit. Simplify answers when possible and follow directions carefully on each problem.

1. Let $f(x) = x^2 - x$.

   (a) (4 points) *Approximate* the area under $f(x)$ on $[0, 4]$ using four rectangles and using left hand endpoints.

   First, notice that since $a = 0$, $b = 4$, and $n = 4$, then $\Delta x = \frac{b-a}{n} = \frac{4-0}{4} = 1$. Also note that we are using left-hand endpoints.

   Therefore, $A \approx \sum_{k=1}^{4} f(x_k)\Delta x = f(0)\cdot(1) + f(1)\cdot(1) + f(2)\cdot(1) + f(3)\cdot 1 = 0(1) + 0(1) + 2(1) + 6(1) = 8\,\text{units}^2$.

   (b) (5 points) Use summation notation to write an expression that represents approximating the area under $f(x)$ on $[0, 4]$ using n rectangles with right hand endpoints.

   In a general equally spaced right-handed sum, $\Delta x = \frac{b-a}{n} = \frac{4-0}{n} = \frac{4}{n}$ and $x_k = 0 + k\Delta x = \frac{4k}{n}$.

   Then $A \approx \sum_{k=1}^{n} f\left(\frac{4k}{n}\right)\left(\frac{4}{n}\right) = \sum_{k=1}^{n}\left[\left(\frac{4k}{n}\right)^2 - \left(\frac{4k}{n}\right)\right]\left(\frac{4}{n}\right) = \sum_{k=1}^{n}\left(\frac{16k^2}{n^2} - \frac{4k}{n}\right)\left(\frac{4}{n}\right) = \sum_{k=1}^{n}\left(\frac{64k^2}{n^3} - \frac{16k}{n^2}\right)$

   (c) (7 points) Use the expression you found in part (b) to find the *exact area* under $f(x)$ on $[0, 4]$ by first using summation formulas and then taking the limit as $n \to \infty$.

   Using the result from above, $A = \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{64k^2}{n^3} - \frac{16k}{n^2}\right) = \lim_{n\to\infty}\left(\frac{64}{n^3}\sum_{k=1}^{n}k^2 - \frac{16}{n^2}\sum_{k=1}^{n}k\right)$

   $$= \lim_{n\to\infty}\left(\frac{64}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] - \frac{16}{n^2}\left[\frac{n(n+1)}{2}\right]\right) = \lim_{n\to\infty}\left(\frac{64}{n^3}\left(\frac{2n^3+3n^2+n}{6}\right) - \frac{16}{n^2}\left(\frac{n^2+n}{2}\right)\right)$$

   $$= \lim_{n\to\infty}\left(\frac{64}{6}\cdot\frac{2n^3+3n^2+n}{n^3} - \frac{16}{2}\cdot\frac{n^2+n}{n^2}\right) = \frac{64}{6}\cdot\frac{2}{1} - \frac{16}{2}\cdot\frac{1}{1} = \frac{64}{3} - 8 = \frac{64}{3} - \frac{24}{3} = \frac{40}{3}\,\text{units}^2.$$

   (d) (4 points) Verify your answer to part (c) by using the Fundamental Theorem of Calculus to evaluate $\int_0^4 f(x)\,dx$.

   $$\int_0^4 x^2 - x\,dx = \frac{1}{3}x^3 - \frac{1}{2}x^2\Big|_0^4 = \frac{1}{3}(4)^3 - \frac{1}{2}(4)^2 - (0-0) = \frac{64}{3} - 8 = \frac{64}{3} - \frac{24}{3} = \frac{40}{3}\,\text{units}^2.$$

2. (4 points each) Compute the following:

(a) $\frac{d}{dx}\left(\int_0^{x^4} \frac{1}{\sqrt{t^2+7}}\,dt\right)$

$$= \frac{d}{dx}\left(F(x^4) - F(0)\right)$$
$$= F'(x^4)\frac{d}{dx}\left(x^4\right) - 0$$
$$= f(x^4)\left(4x^3\right)$$
$$= \frac{4x^3}{\sqrt{x^8+7}}$$

(b) $\int_0^3 \frac{d}{dt}\frac{1}{\sqrt{t^2+7}}\,dt$

$$= F(2) - F(0)$$
$$= \frac{1}{\sqrt{9+7}} - \frac{1}{\sqrt{0+7}}$$
$$= \frac{1}{4} - \frac{1}{\sqrt{7}}$$
$$= \frac{\sqrt{7}-4}{4\sqrt{7}} = \frac{7-4\sqrt{7}}{28}$$

3. (5 points) State the Second Part of the Fundamental Theorem of Calculus.

If f is continuous at every point in $[a,b]$ and F is any antiderivative of f on $[a,b]$, then $\int_a^b f(x)\ dx = F(b) - F(a)$

4. (4 points each) Suppose that $f(x)$ is a continuous function satisfying: $\int_1^5 f(x)\,dx = 7$, $\int_5^8 f(x)\,dx = -2$, and $\int_5^8 g(x)\,dx = 3$. Find:

(a) $\int_1^8 f(x)\,dx$

$$= \int_1^5 f(x)\,dx + \int_5^8 f(x)\,dx$$
$$= 7 + (-2) = 5$$

(b) $\int_5^8 2f(x) - 3g(x)\,dx$

$$= 2\int_5^8 f(x)\,dx - 3\int_5^8 g(x)\,dx$$
$$= 2(-2) - 3(3) = -4 - 9 = -13$$

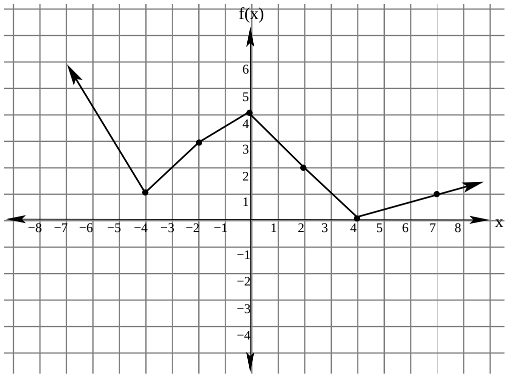
(c) $\int_5^1 f(x)\,dx$

$$= -\int_1^5 f(x)\,dx$$
$$= -7$$

(d) the average value of $g(x)$ on $[5,8]$

$$= \frac{1}{8-5}\int_5^8 g(x)\,dx$$
$$= \tfrac{1}{3}\,(3) = 1$$

5. Given the following graph of $f(x)$, let $g(x) = \int_{-2}^{x} f(t)\,dt$.



(a) (4 points) Find $g(2)$.

$$g(3) = \int_{-2}^{2} f(t)\,dt.$$

$= \text{Area under } f \text{ on } [-2, 2]$

$= \frac{1}{2}(3+4)(2) + \frac{1}{2}(4+2)(2)$

$= 7 + 6 = 13\,\text{units}^2$

(b) (4 points) Find $g'(2)$

$g'(2) = f(2) = 2.$

(c) (4 points) Find $g''(2)$

$g''(2) = f'(2) = \frac{2-0}{2-4} = -1.$

6. Evaluate each of the following (if possible):

(a) (5 points) $\int \frac{x-2}{x^4}\,dx$

$$= \int \frac{x}{x^4} - \frac{2}{x^4}\,dx$$

$$= \int x^{-3} - 2x^{-4}\,dx$$

$$= -\frac{1}{2}x^{-2} - 2\left(\frac{1}{-3}\right)x^{-3} + C$$

$$= -\frac{1}{2x^2} + \frac{2}{3x^3} + C$$

(b) (6 points) $\int \frac{2x}{\sqrt{3x^2-1}}\,dx$

Let $u = 3x^2 - 1$. Then $du = 6x\,dx$, or $\frac{1}{3}\,du = 2x\,dx$

Then we have $\int \frac{1}{3}u^{-\frac{1}{2}}\,du$

$= \frac{1}{3} \cdot 2u^{\frac{1}{2}} + C = \frac{2}{3}\sqrt{3x^2-1} + C$

(c) (5 points) $\int_0^1 \frac{x-4}{x^4}\,dx$

Notice that the integrand $\frac{x-4}{x^4}$ has an infinite discontinuity at $x = 0$.

Therefore, the integrand is not integrable on $[0, 1]$, so there is no solution.

(d) (8 points) $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 3\sin\theta\cos^2\theta\,d\theta$

Let $u = \cos\theta$. Then $du = -\sin\theta\,d\theta$. Also, $u\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$, and $u\left(\frac{\pi}{2}\right) = 0$

Then we have $\int_{\frac{\sqrt{3}}{2}}^{0} -3u^2\,du = \int_0^{\frac{\sqrt{3}}{2}} 3u^2\,du$

$$= u^3\Big|_0^{\frac{\sqrt{3}}{2}} = \left(\frac{\sqrt{3}}{2}\right)^3 - 0 = \frac{3\sqrt{3}}{8}\,\text{units}^2$$

7. (5 points each)

(a) Use the Trapezoidal Rule with $n = 4$ to approximate $\int_{-2}^{2} \left(5x^2 - x^3\right)\, dx$

Notice that $n = 4$, so $\Delta x = \frac{b-a}{n} = \frac{2-(-2)}{4} = \frac{4}{4} = 1$.
Recall that in the Trapezoidal Rule, $A \approx \frac{b-a}{2n}\left(f(a) + 2f(a+\Delta x) + ... + 2f(a+(k-1)\Delta x) + f(x_n)\right)$
$= \frac{2-(-2)}{2\cdot 4}\left(f(-2) + 2f(-1) + 2f(0) + 2f(1) + f(2)\right) = \frac{1}{2}\left(28 + 2(6) + 2(0) + 2(4) + 12\right)$
$= \frac{1}{2}[60] = 30$ units^2.

(b) Find the maximum possible error for your approximation from part (a). Recall that $E_n \le \frac{M(b-a)^3}{12n^2}$

Recall that the maximum error for the Trapezoidal Rule is given by $Error \le \frac{M(b-a)^3}{12n^2}$ where M is the maximum absolute value of the second derivative $f''(x)$ in the interval $[a, b]$.

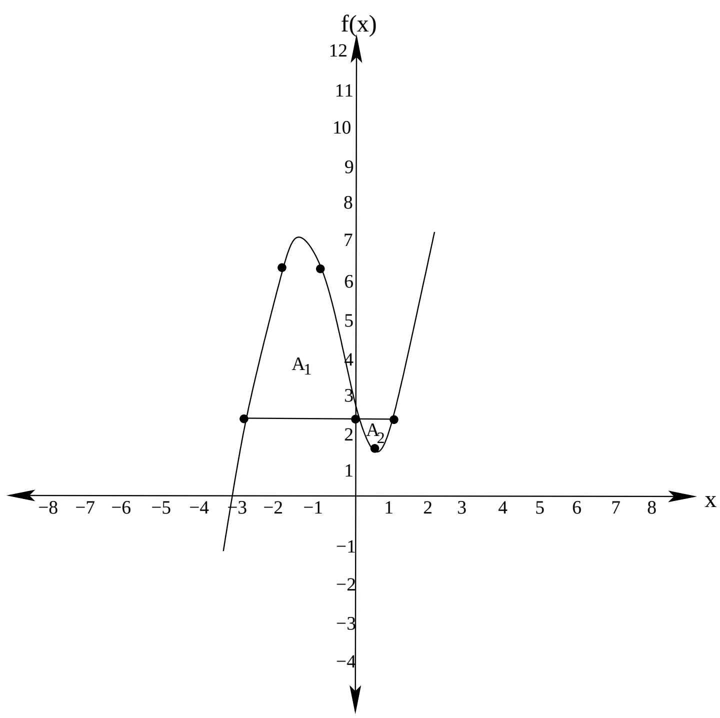
In this case, $f'(x) = 10x - 3x^2$ and $f''(x) = 10 - 6x$. Since $f''(x)$ is linear with positive slope, it had no critical points, so it's extrema occur at the endpoints of the interval. Notice that $f''(-2) = 10 + 12 = 22$ and $f''(2) = 10 - 12 = -2$, so $M = 22$ and $n = 4$

Therefore, $E_n \le \frac{22(4)^3}{12(4)^2} = \frac{22(4)}{12} = \frac{22}{3}$.

8. (10 points) Set up an integral that can be used to find the area bounded between $y = x^3 + 2x^2 - 3x + 2$ and $y = 2$. You DO NOT need to evaluate this integral.

First, note that if $x^3 + 2x^2 - 3x + 2 = 2$, then $x^3 + 2x^2 - 3x = 0$, so $x(x^2 + 2x - 3) = 0$, or $x(x+3)(x-1) = 0$. Hence $x = 0$, $x = -3$, or $x = 1$.

Next, we graph the functions involved in defining our region. Note that the points of intersection are $(-3, 2), (0, 2)$, and $(1, 2)$. Also, note that when $x = -1$ we have $(-1, 6)$ on the cubic. Similarly, when $x = \frac{1}{2}$ we have $\left(\frac{1}{2}, \frac{9}{8}\right)$ on the cubic. $y = 2$ is a horizontal line.



From this, we set up two integrals whose sum gives the total area:
$A = \int_{-3}^{0} (x^3 + 2x^2 - 3x + 2) - (2)\, dx + \int_{0}^{1} (2) - (x^3 + 2x^2 - 3x + 2)\, dx.$