Math 261
Extrema of Functions

Definitions: Let $f(x)$ be a function defined on an open interval I and let x_1 and x_2 be values in I.

- f is **increasing** on I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.
- f is **decreasing** on I if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.
- f is **constant** on I if $f(x_1) = f(x_2)$ for all x_1, x_2 in I.

Now, suppose that $f(x)$ is a function defined on a set of real numbers S and let c be a value in S.

- $f(c)$ is the **maximum value** of f on S if $f(x) \leq f(c)$ for all x in the set S.
- $f(c)$ is the **minimum value** of f on S if $f(x) \geq f(c)$ for all x in the set S.

Taken together, these values are called the **extreme values** or **extrema** of $f(x)$ on the set S. Note that the extreme values may occur more than once.

Examples:

1. Consider $f(x) = 4 - x^2$ where S is the interval $[-2, 2]$.

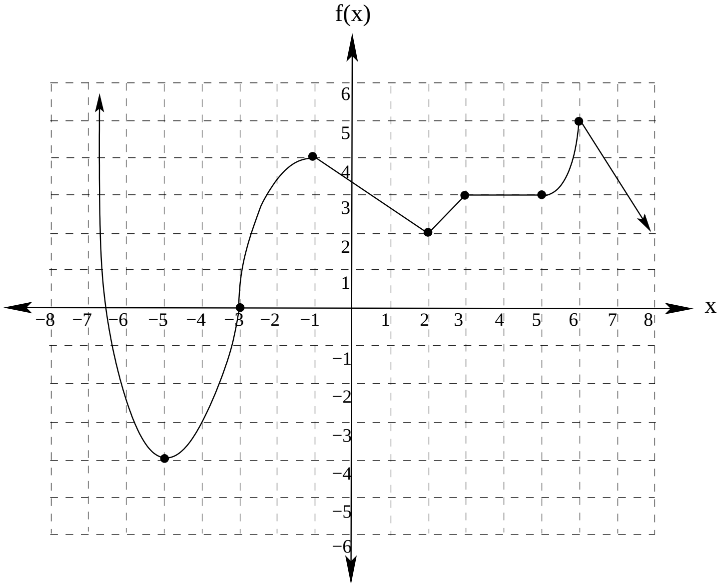
2. Consider $f(x) = 3\cos x$ where S is the interval $[0, 4\pi]$.

Theorem 1: The Extreme Value Theorem: Let $f(x)$ be a function that is continuous on a closed interval $[a, b]$. Then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m$ and $f(x_2) = M$, and $m \leq f(x) \leq M$ for every x in $[a, b]$.

Definitions: Let $f(x)$ be a function and suppose that c is in the domain of f.

- $f(c)$ is a **local maximum** if there is an open interval (a, b) containing c such that $f(x) \leq f(c)$ for every x in (a, b).
- $f(c)$ is a **local minimum** if there is an open interval (a, b) containing c such that $f(x) \geq f(c)$ for every x in (a, b).

Example: Consider the following graph. Find and classify all local extrema.



Theorem 2: If a function f has a local extremum at a number c in an open interval, then either $f'(c) = 0$, or $f'(c)$ does not exist.

Corollary: If $f'(c)$ exists and $f'(c) \neq 0$, then $f(c)$ is not a local extremum of the function $f(x)$.

Theorem: If a function $f(x)$ is continuous on a closed interval $[a, b]$ and an extremum of f occurs at a number c in the open interval (a, b), then either $f'(c) = 0$ or $f'(c)$ does not exist.

Definition: A number c in the domain of a function $f(x)$ is a **critical number** of f if either $f'(c) = 0$ or $f'(c)$ does not exist.

Note: Every local extremum of a function $f(x)$ occurs at a critical number. However, a specific critical number may or may not be the location of a local extremum.

A Method For Finding the Extrema of a Function on a Closed Interval:

1. Check to see whether or not $f(x)$ is continuous on the interval $[a, b]$.
2. Differentiate f and find all critical numbers of f with in (a, b).
3. Evaluate to find $f(c)$ for every critical number c in (a, b).
4. Evaluate f at both endpoints (find $f(a)$ and $f(b)$).
5. Compare the values found in steps 3 and 4. The largest value is the absolute maximum and the smallest value is the absolute minimum.

Example: Let $f(x) = x^4 - 2x^2 + 17$. Find the absolute extrema of f on $[-2, 2]$.