

**The Intermediate Value Theorem:** If a function  $f$  is continuous on a closed interval  $[a, b]$  and  $w$  is any number between  $f(a)$  and  $f(b)$ , then there is at least one number  $c$  in  $[a, b]$  such that  $f(c) = w$ .

- *Intuitively*, this means that a continuous function  $f$  on an closed interval  $[a, b]$  attains every  $y$  value in between its boundary values ( $f(a)$  and  $f(b)$ ) at least once in the interval  $[a, b]$ .
- We can use the IVT to show that a function has a certain value within a given interval. More specifically, if a continuous function is positive at one point and negative at another, then it must be zero somewhere in between.

**The Extreme Value Theorem:** If a function  $f$  is continuous on a closed interval  $[a, b]$ , then  $f$  takes on a minimum and maximum value at least once in  $[a, b]$ .

- *Intuitively*, the idea is that a continuous function in a closed interval must have extrema. Since continuous functions have no jumps or gaps, there must be a highest value and a least value of the function on any closed interval.
- In practice, we know that these extrema must occur either at a critical point or an end point (that is, at the top of a “hill”, at the bottom of a “valley”, or on the way to a more extreme value when the boundary of the interval “gets in the way”). Therefore, we find the extrema of a continuous function on closed interval by:

1. Finding all critical numbers
2. Computing the value of the function on all critical numbers **inside** the interval in question
3. Computing the value of the function at the boundary points of the interval
4. Compare all these values: the biggest value is the max, the smallest is the min.

**Rolle's Theorem:** If a function  $f$  is continuous on a closed interval  $[a, b]$ , differentiable on the open interval  $(a, b)$ , and  $f(a) = f(b)$ , then  $f'(c) = 0$  for at least one number  $c$  in  $(a, b)$ .

- *Intuitively*, this theorem says that if a differentiable function attains the same value twice, then it must have a “turning point” somewhere in between.

**The Mean Value Theorem:** If a function  $f$  is continuous on a closed interval  $[a, b]$ , differentiable on the open interval  $(a, b)$ , then there exists a number  $c$  in  $(a, b)$  such that  $f'(c) = \frac{f(b) - f(a)}{b - a}$ . Or, in other words, where  $f(b) - f(a) = f'(c)(b - a)$

- *Intuitively*, this theorem says that if we pick any pair of points on a differentiable function and compute the slope of the secant line between those two points, then there is a point  $c$  in between whose tangent line slope is the same as the slope of the secant line.

**Example:**

Let  $f(x) = x^3 - 3x$ . Notice that  $f$  is continuous since it is a polynomial, and since  $f'(x) = 3x^2 - 3$  is also a polynomial, then  $f$  is also differentiable.

1. To see how the IVT applies to this function, notice that  $f(-1) = -1 + 3 = 2$  while  $f(1) = 1 - 3 = -2$ , so there must be a  $c$  between  $-1$  and  $2$  such that  $f(c) = 0$ . Of course it is easy to see where the root is, since we can easily notice and verify that  $f(0) = 0$ .
2. To see how the EVT can be applied to this function, let's consider  $f$  on the interval  $[0, 3]$ . Since  $f'(x) = 3x^2 - 3$ , the critical numbers of  $f(x)$  occur when  $3x^2 - 3 = 0$ , or  $3x^2 = 3$ . That is, when  $x^2 = 1$ , or  $x = \pm 1$ . However, notice that  $x = -1$  is outside the interval we are considering, so to find the extrema of  $f$  on  $[0, 3]$ , we only check the values of  $f$  when  $x = 0, 1, 3$ .

Notice that  $f(0) = 0$ ,  $f(1) = -2$ , and  $f(3) = 27 - 9 = 18$ . Therefore, on this interval, the maximum value of  $f$  is 18, and the minimum value of  $f$  is  $-2$ .

3. To see one way that Rolle's Theorem can be applied to this function, notice that  $f(x) = 0$  when  $x^3 - 3x = 0$ , or when  $x(x^2 - 3) = 0$ . That is, when  $x = 0$ , and when  $x = \pm\sqrt{3}$ . Therefore, according to Rolle's Theorem, there must be an  $x$ -value between  $-\sqrt{3}$  and zero where  $f'(x) = 0$ , and there must also be an  $x$ -value between zero and  $\sqrt{3}$  where  $f'(x) = 0$ . [This does in fact end up being true, since we have already seen that  $f'(-1) = 0$ , and  $f'(1) = 0$ ].

4. To see how the Mean Value Theorem applies to this function, consider the function  $f$  on the interval  $[0, 2]$ . Notice that  $f(0) = 0$ , and  $f(2) = 2^3 - 3(2) = 8 - 6 = 2$ , so the endpoints of this interval are  $(0, 0)$  and  $(2, 2)$ . Therefore, the slope of the secant line between these endpoints is:  $m_{sec} = \frac{2-0}{2-0} = \frac{2}{2} = 1$ . The MVT claims that there should be at least one  $x$ -value  $c$  between 0 and 2 for which  $f'(c) = 1$ . Let's find it:

Suppose  $f'(c) = 1$ . Then  $3c^2 - 3 = 1$ , so  $3c^2 = 4$ . Therefore,  $c^2 = \frac{4}{3}$ , so  $c = \pm\sqrt{\frac{4}{3}} = \pm\frac{2}{\sqrt{3}} = \pm\frac{2\sqrt{3}}{3}$ .

Notice that  $\frac{2\sqrt{3}}{3} \approx 1.1547$ , so we have found a  $c$  value in the predicted interval.

**Note:** On our next lab, we will also look at how to Apply these theorems when we only have a table of values for our function and its derivative rather than actual equations for our functions.