Math 261
Newton's Method Handout

Example: Let $f(x) = 2x^3 - 6x + 1$. Notice that $f(x)$ is a continuous function with $f(0) = 1$ and $f(1) = 2 - 6 + 1 = -3$. Therefore, by the Intermediate Value Theorem, $f(x)$ must have a zero somewhere between 0 and 1. Our goal is to approximate this zero.

To sketch the graph of $f(x)$, note that $f'(x) = 6x^2 - 6$, which has critical numbers when $6x^2 = 6$, or $x^2 = 1$. That is, when $x = \pm 1$.

This leads to the sign chart:

So $f(x)$ is increasing on $(-\infty, -1] \cup [1, \infty)$ and decreasing on $[-1, 1]$.

Similarly, $f''(x) = 12x$, which has one critical value, $x = 0$. Notice that $f''(x) < 0$ when $x < 0$ and $f''(x) > 0$ when $x > 0$.

If we also note that $f(-1) = 5$ is a local max and $f(1) = -3$ is a local min, then we have the following graph for $f(x)$:

To find the zero of $f(x)$ on $[0, 1]$, we begin with an initial guess of $x = \frac{1}{2}$ and consider the tangent line to $f(x)$ at this point.

Notice that $f'\left(\frac{1}{2}\right) = 6\left(\frac{1}{2}\right)^2 - 6 = \frac{6}{4} - 6 = \frac{3}{2} - \frac{12}{2} = -\frac{9}{2}$.

Also, $f\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^3 - 6\left(\frac{1}{2}\right) + 1 = \frac{1}{4} - 3 + 1 = -\frac{7}{4}$.

Therefore, an equation for the tangent line to $f(x)$ at the point $\left(\frac{1}{2}, -\frac{7}{4}\right)$ is $y + \frac{7}{5} = -\frac{9}{2}\left(x - \frac{1}{2}\right)$ or $y = -\frac{9}{2}x + \frac{9}{4} - \frac{7}{4}$. Which, in slope intercept form is: $y = -\frac{9}{2}x + \frac{1}{2}$.

We now find the x-intercept of this line: If $y = 0$, $\frac{9}{2}x = \frac{1}{2}$, so $x = \frac{1}{2} \cdot \frac{2}{9} = \frac{1}{9}$.

We claim that if we set $x_1 = \frac{1}{9}$, then x_1 is a better approximation of the zero of $f(x)$ in the interval $[0, 1]$.

What if we repeated this process to find x_2 based on our previous approximation $x_1 = \frac{1}{9}$? We claim that this would get a still better approximation of the zero e are looking for.

However, having to redo similar computations several times is both tedious and unnecessary. We can develop a procedure that will help us find our next guess more directly as follows:

Suppose that $y - f(x_n) = f'(x_n)(x - x_n)$ is a tangent line to $f(x)$ at the point associated to the approximation x_n, and suppose that this tangent line has x-intercept $(x_{n+1}, 0)$. Then, setting $x = x_{n+1}$ and $y = 0$, we have:

$-f(x_n) = f'(x_n)(x_{n+1} - x_n)$, or $-f(x_n) = f'(x_n)x_{n+1} - f'(x_n)x_n$.

Therefore, $-f(x_n) + f'(x_n)x_n = f'(x_n)x_{n+1}$.

Hence $\frac{-f(x_n)+f'(x_n)x_n}{f'(x_n)} = x_{n+1}$

That is, $x_{n+1} = \frac{-f(x_n)}{f'(x_n)} + \frac{f'(x_n)x_n}{f'(x_n)}$.

Thus $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.

This gives us the framework we need to carry out Newton's Method.

Newton's Method:

1. Let x_0 be an initial guess about the value of a root of a differentiable function $f(x)$ (a solution to the equation $f(x) = 0$).

2. Use the following formula to find additional approximations:

   $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.

Continuing our example from above: If $f(x) = 2x^3 - 6x + 1$, then $f'(x) = 6x^2 - 6$. We took $x_0 = \frac{1}{2}$.

Using the formula for Newton's method, we see that $x_1 = \frac{1}{2} - \frac{f\left(\frac{1}{2}\right)}{f'\left(\frac{1}{2}\right)} = \frac{1}{2} - \frac{-\frac{7}{4}}{-\frac{9}{2}} = \frac{1}{2} - \frac{7}{4} \cdot \frac{2}{9} = \frac{1}{2} - \frac{7}{18} = \frac{9}{28} - \frac{7}{18} = \frac{2}{18} = \frac{1}{9}$.

Continuing in this way, we find that $x_2 \approx 0.167824$, $x_3 \approx 0.167824$ and $x_4 \approx 0.168254$.

Example: Use a polynomial function and Newton's Method to approximate $\sqrt[3]{5}$.

To find a function $f(x)$ that has $\sqrt[3]{5}$ as a root, suppose $x - \sqrt[3]{5} = 0$. Then $x = \sqrt[3]{5}$, so $x^3 = 5$. Therefore, let $f(x) = x^3 - 5$. Notice that $f'(x) = 3x^2$. Since we know that $1 = \sqrt[3]{1} < \sqrt[3]{5} < \sqrt[3]{8} = 2$, we take $x_0 = 2$ as our initial guess.

To complete this example, use Newton's Method to find x_1, x_2, ... x_5 (How do we know when to stop?).