Section 4.6: Adaptive Quadrature Example

Example: Note that $\int_0^{1.5} \tan x\, dx = -\ln|\cos x|\Big|_0^{1.5} \approx 2.648783654$.
We will demonstrate the method of adaptive quadrature. Given $f(x) = \tan x$, we begin by taking $h = 0.75$.
Our goal is to approximate $\int_0^{1.5} \tan x\, dx$ to within an error tolerance of $\varepsilon = 0.01$.

Step 1: $S(0, 1.5) = \frac{0.75}{3}[f(0) + 4f(0.75) + f(1.5)] \approx 4.456951447$
$S(0, 0.75) = \frac{0.75}{6}[f(0) + 4f(0.375) + f(0.75)] \approx 0.313262845$
$S(0.75, 1.5) = \frac{0.75}{6}[f(0.75) + 4f(1.125) + f(1.5)] \approx 2.925412689$.
Thus $|S(0, 1.5) - S(0, 0.75) - S(0.75, 1.5)| \approx 1.218276$, which is **not** less than $15\varepsilon = 0.15$. Hence we need to subdivide both half intervals and compute again.

Step 2a: (Left Half) $S(0, 0.375) = \frac{0.75}{12}[f(0) + 4f(0.1875) + f(0.375)] \approx 0.0720338137$
$S(0.375, 0.75) = \frac{0.75}{12}[f(0.375) + 4f(0.5625) + f(0.75)] \approx .2404358582$.
Thus $|S(0, 0.75) - S(0, 0.375) - S(0.375, 0.75)| \approx 0.000793173$, which **is** less than $15 \cdot \frac{\varepsilon}{2} = 0.075$. Hence we **do not** need to subdivide this half interval a second time.

Step 2b: (Right Half) $S(0.75, 1.125) = \frac{0.75}{12}[f(0.75) + 4f(0.9375) + f(1.125)] \approx 0.5295284776$
$S(1.125, 1.50) = \frac{0.75}{12}[f(1.125) + 4f(1.3125) + f(1.50)] \approx 1.958383993$.
Thus $|S(0.75, 1.50) - S(0.75, 1.125) - S(1.125, 1.50)| \approx 0.4375...$, which is **not** less than $15 \cdot \frac{\varepsilon}{2} = 0.075$. Hence we need to subdivide this half interval again.

Step 3a: (Left Half) $S(0.75, 0.9375) = \frac{0.75}{24}[f(0.75) + 4f(0.84375) + f(0.9375)] \approx 0.21218778$
$S(0.9375, 1.125) = \frac{0.75}{24}[f(0.9375) + 4f(1.03125) + f(1.125)] \approx 0.316703867$.
Thus $|S(0.75, 1.125) - S(0.75, 0.9375) - S(0.9375, 1.125)| \approx 0.000636$, which **is** less than $15 \cdot \frac{\varepsilon}{4} = 0.0375$. Hence we **do not** need to subdivide this half interval again.

Step 3b: (Right Half) $S(1.125, 1.3125) = \frac{0.75}{24}[f(1.125) + 4f(1.21875) + f(1.3125)] \approx 0.523950957$
$S(1.3125, 1.50) = \frac{0.75}{24}[f(1.3125) + 4f(1.40625) + f(1.50)] \approx 1.311747793$.
Thus $|S(1.125, 1.50) - S(1.125, 1.3125) - S(1.3125, 1.50)| \approx 0.122...$, which **is not** less than $15 \cdot \frac{\varepsilon}{4} = 0.0375$. Hence we do need to subdivide this half interval again.

Step 4a: (Left Half) $S(1.125, 1.21875) = \frac{0.75}{48}[f(1.125) + 4f(1.171875) + f(1.21875)] \approx 0.223502989$
$S(1.21875, 1.3125) = \frac{0.75}{48}[f(1.21875) + 4f(1.265625) + f(1.3125)] \approx 0.30081106$.
Thus $|S(1.125, 1.3125) - S(1.125, 1.21875) - S(1.21875, 1.3125)| \approx 0.00036...$, which **is** less than $15 \cdot \frac{\varepsilon}{8} = 0.01875$. Hence we **do not** need to subdivide this half interval again.

Step 4b: (Right Half) $S(1.3125, 1.40625) = \frac{0.75}{48}[f(1.3125) + 4f(1.359375) + f(1.40625)] \approx 0.4441432269$
$S(1.40625, 1.50) = \frac{0.75}{48}[f(1.40625) + 4f(1.453125) + f(1.50)] \approx 0.8431208694$.
Thus $|S(1.3125, 1.50) - S(1.3125, 1.40625) - S(1.40625, 1.50)| \approx 0.0244...$, which **is not** less than $15 \cdot \frac{\varepsilon}{8} = 0.01875$. Hence we need to subdivide this half interval again.

Step 5a: (Left Half) $S(1.3125, 1.359375) = \frac{0.75}{96}[f(1.3125) + 4f(1.3359375) + f(1.359375)] \approx 0.196573833$
$S(1.359375, 1.40625) = \frac{0.75}{96}[f(1.359375) + 4f(1.3828125) + f(1.40625)] \approx 0.247724715$.
Thus $|S(1.3125, 1.40625) - S(1.3125, 1.359375) - S(1.359375, 1.40625)| \approx 0.00014...$, which **is** less than $15 \cdot \frac{\varepsilon}{16} = 0.009375$. Hence we **do not** need to subdivide this half interval again.

Step 5b: (Right Half) $S(1.40625, 1.453125) = \frac{0.75}{96}[f(1.40625) + 4f(1.4296875) + f(1.453125)] \approx 0.333124023$
$S(1.453125, 1.50) = \frac{0.75}{96}[f(1.453125) + 4f(1.4765625) + f(1.50)] \approx 0.506892887$.
Thus $|S(1.40625, 1.50) - S(1.40625, 1.453125) - S(1.453125, 1.50)| \approx 0.0031...$, which **is** less than $15 \cdot \frac{\varepsilon}{16} = 0.009375$. Hence we **do not** need to subdivide this half interval again, so we are done computing half intervals.

Step 6: To finish, we add up the approximation associated with each previously computed interval:

$\int_0^{1.5} \tan x\, dx \approx S(0, 0.375) + S(0.375, 0.75) + S(0.75, 0.9375) + S(0.9375, 1.125) + S(1.125, 1.21875) + S(1.21875, 1.3125) + S(1.3125, 1.359375) + S(1.359375, 1.40625) + S(1.40625, 1.453125) + S(1.453125, 1.5)$
$\approx 0.072033814 + 0.2404358582 + 0.21218778 + 0.316703867 + 0.223502989 + 0.30081106 + 0.196573833 + 0.247724715 + 0.333124023 + 0.506892887$
$= 2.649260871$ (Notice that this does approximate the orginal definite itegral to within our desired error tolerance).