Math 450

Section 3.4: Hermite Interpolation

Main Idea: The Lagrange interpolating polynomial, $P_n(x)$, has been defined so that the polynomial agrees with the original function $f(x)$ at $n+1$ distinct input values $x_0, x_1, \cdots, x_n$. On the other hand, Taylor polynomials approximate a function using a *single* center point at which we know the value of the function **and** the value of several derivatives. Our goal is to generalize both the Lagrange polynomial and the Taylor polynomial by forming an interpolating polynomial that agrees with the function *both* at several distinct points *and* at a given number of derivatives of the function at those distinct points. A polynomial that satisfies these conditions is called an *osculating polynomial.*

Definition: Assume $x_0, x_1, \cdots, x_n \in [a,b]$ are $n+1$ distinct numbers. Also, assume $m_0, m_1, \cdots, m_n$ are nonnegative integers where each integer, m_i, corresponds with x_i. Further assume $f \in C^m[a,b]$ where $m = max\{m_i : 0 \le i \le n\}$. The **osculating polynomial** that approximates f is the polynomial $P(x)$ of least degree such that $\frac{d^k P(x_i)}{dx^k} = \frac{d^k f(x_i)}{dx^k}$ for each $i = 0, 1, \cdots n$ and $k = 0, 1, \cdots, m_i$. That is:

$f(x_0) = P(x_0), f'(x_0) = P'(x_0), f''(x_0) = P''(x_0), \cdots f^{(m_0)}(x_0) = P^{(m_0)}(x_0),$

$f(x_1) = P(x_1), f'(x_1) = P'(x_1), f''(x_0) = P''(x_1), \cdots f^{(m_1)}(x_1) = P^{(m_1)}(x_1),$

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$f(x_n) = P(x_n), f'(x_n) = P'(x_n), f''(x_n) = P''(x_n), \cdots f^{(m_n)}(x_n) = P^{(m_n)}(x_n).$

Note: When there is a single point, x_0, the osculating polynomial approximating f is the Taylor polynomial of m_0th degree.

Definition: The osculating polynomial of f formed when $m_0 = m_1 = \cdots = m_n = 1$ is called the `Hermite polynomial`.

Note: The graph of the Hermite polynomial of f agrees with f at $n+1$ distinct points and has the same tangent lines as f at those $n+1$ distinct points.

Recall: The Lagrange coefficient polynomial is defined by:

$$L_{n,k} = \frac{(x-x_0)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}$$

Also, $L_{n,k}(x_i) = \begin{cases} 0 & \text{if } i \neq k \\ 1 & \text{if } i = k \end{cases}$

Theorem: Assume $f \in C^1[a,b]$ and $x_0, x_1, \cdots, x_n \in [,b]$ are distinct points. Then the unique polynomial of degree less than or equal to $2n+1$ is given by:

$$H_{2n+1}(x) = \sum_{j=0}^{n} f(x_j) H_{n,j}(x) + \sum_{j=0}^{n} f'(x_j) \hat{H}_{n,j}(x)$$

where

$$H_{n,j}(x) = \left[1 - 2(x-x_j)L'_{n,j}(x_j)\right] L^2_{n,j}(x)$$

and

$$\hat{H}_{n,j}(x) = (x-x_j)L^2_{n,j}(x)$$

Example: Suppose that $f(0) = 2, f'(0) = 1, f(1) = 4, f'(1) = -1, f(3) = 5, f'(3) = -2$. Find the Hermite interpolating polynomial and use it to approximate the value of $f(2)$.

Solution:

$L_{2,0}(x) = \frac{(x-1)(x-3)}{(0-1)(0-3)} = \frac{1}{3}x^2 - \frac{4}{3}x + 1$. Therefore, $L'_{2,0}(x) = \frac{2}{3}x - \frac{4}{3}$.
$L_{2,1}(x) = \frac{(x-0)(x-3)}{(1-0)(1-3)} = -\frac{1}{2}x^2 + \frac{3}{2}x$. Therefore, $L'_{2,1}(x) = -x + \frac{3}{2}$.
$L_{2,2}(x) = \frac{(x-0)(x-1)}{(3-0)(3-1)} = \frac{1}{6}x^2 - \frac{1}{6}x$. Therefore, $L'_{2,2}(x) = \frac{1}{3}x - \frac{1}{6}$.

$H_{2,0}(x) = \left[1 - 2(x-0)\left(\frac{2}{3}(0) - \frac{4}{3}\right)\right]\left[\frac{1}{3}x^2 - \frac{4}{3}x + 1\right]^2 = \left(1 + \frac{8}{3}x\right)\left[\frac{1}{3}x^2 - \frac{4}{3}x + 1\right]^2$
$H_{2,1}(x) = \left[1 - 2(x-1)\left(-1 + \frac{3}{2}\right)\right]\left[-\frac{1}{2}x^2 + \frac{3}{2}x\right]^2 = (2-x)\left[-\frac{1}{2}x^2 + \frac{3}{2}x\right]^2$
$H_{2,2}(x) = \left[1 - 2(x-3)\left(\frac{1}{3}(3) - \frac{1}{6}\right)\right]\left[\frac{1}{6}x^2 - \frac{1}{6}x\right]^2 = \left(6 - \frac{5}{3}x\right)\left[\frac{1}{6}x^2 - \frac{1}{6}x\right]^2$
$\hat{H}_{2,0}(x) = (x-0)\left[\frac{1}{3}x^2 - \frac{4}{3}x + 1\right]^2$

$\hat{H}_{2,1}(x) = (x-1)\left[-\frac{1}{2}x^2 + \frac{3}{2}x\right]^2$
$\hat{H}_{2,2}(x) = (x-3)\left[\frac{1}{6}x^2 - \frac{1}{6}x\right]^2$

$H_5(x) = 2\left(1 + \frac{8}{3}x\right)\left[\frac{1}{3}x^2 - \frac{4}{3}x + 1\right]^2 + 4(2-x)\left[-\frac{1}{2}x^2 + \frac{3}{2}x\right]^2 + 5\left(6 - \frac{5}{3}x\right)\left[\frac{1}{6}x^2 - \frac{1}{6}x\right]^2 + 1(x)\left[\frac{1}{3}x^2 - \frac{4}{3}x + 1\right]^2 - 1(x-1)\left[-\frac{1}{2}x^2 + \frac{3}{2}x\right]^2 - 2(x-3)\left[\frac{1}{6}x^2 - \frac{1}{6}x\right]^2$

$H_5(x) = -\frac{5}{6}x^5 + \frac{71}{12}x^4 - \frac{40}{3}x^3 + \frac{37}{4}x^2 + x + 2$

Using this, $f(2) \approx H_5(2) = \frac{7}{3}$

Note: We also could have used a divided difference table incorporating repeated values and derivatives to find this polynomial: