Math 450

Higher Order Taylor Methods

Recall: The recursion for Higher Order Taylor Methods to approximate solutions to a well posed initial value problem of the form $\frac{dy}{dt} = f(t,y)$, $y(a) = \alpha$ over $D = \{(t,y) : a \le t \le b,\ y \in \mathbb{R}\}$ is given by:
$w_0 = \alpha$
$w_{i+1} = w_i + h \cdot T^{(n)}(t_i, w_i)$ where $T^{(n)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) + \cdots + \frac{h^{n-1}}{n!} f^{(n-1)}(t_i, w_i)$.

Example: Consider the well posed IVP given by $\frac{dy}{dt} = t^3 + t - y$, $0 \le t \le 2$, $y(0) = 1$. Use Taylor's Methods of order 2 and 4 to approximate $y(2)$ using step size $h = 0.5$.

We begin by computing the derivatives necessary to find the recursions for second and fourth order Taylor's Methods.

$y'(t) = f(t,y) = t^3 + t - y$

$y''(t) = f'(t,y) = 3t^2 + 1 - y' = 3t^2 + 1 - (t^3 + t - y) = -t^3 + 3t^2 - t + 1 + y$

$y'''(t)f''(t,y) = -3t^2 + 6t - 1 + y' = -3t^2 + 6t - 1 + t^3 + t - y = t^3 - 3t^2 + 7t - 1 - y$

$y^{(4)} = f'''(t,y) = 3t^2 - 6t + 7 - y' = 3t^2 - 6t + 7 - (t^3 + t - y) = -t^3 + 3t^2 - 7t + 7 + y$

From these, we have the following:

$$T^{(2)}(t_i, w_i) = (t_i^3 + t_i - w_i) + \frac{h}{2}(-t_i^3 + 3t + i^2 - t_i + 1 + w_i) = t_i^3 + t_i - w_i - \frac{1}{2}ht_i^3 + \frac{3}{2}ht_i^2 - \frac{1}{2}ht_i + \frac{1}{2}h + \frac{1}{2}hw_i$$

$$T^{(4)}(t_i, w_i) = (t_i^3 + t_i - w_i) + \frac{h}{2}(-t_i^3 + 3t_i^2 - t_i + 1 + w_i) + \frac{h^2}{6}(t_i^3 - 3t_i^2 + 7t_i - 1 - w_i) + \frac{h^3}{24}(-t_i^3 + 3t_i^2 - 7t_i + 7 + w_i) = t_i^3 + t_i - w_i - \frac{1}{2}ht_i^3 + \frac{3}{2}ht_i^2 - \frac{1}{2}ht_i + \frac{1}{2}h + \frac{1}{2}hw_i + \frac{1}{6}h^2t_i^3 - \frac{1}{2}h^2t_i^2 + \frac{7}{6}h^2t_i - \frac{1}{6}h^2 - \frac{1}{6}h^2w_i - \frac{1}{24}h^3t_i^3 + \frac{1}{8}h^3t_i^2 - \frac{7}{24}h^3t_i + \frac{7}{24}h^3 + \frac{1}{24}h^3w_i$$

Using these, we have the following:

Second Order Taylor:

$w_0 = 1$

$w_{i+1} = w_i + h \cdot T^{(2)}(t_i, w_i) = w_i + (0.5) \cdot \left(t_i^3 + t_i - w_i - \frac{1}{2}(0.5)t_i^3 + \frac{3}{2}(0.5)t_i^2 - \frac{1}{2}(0.5)t_i + \frac{1}{2}(0.5) + \frac{1}{2}(0.5)w_i\right)$
$w_1 = 0.75$

$w_2 = 0.9218750000$

$w_3 = 1.826171875$

$w_4 = 3.938232422$

Fourth Order Taylor:

$w_0 = 1$

$$w_{i+1} = w_i + h \cdot T^{(4)}(t_i, w_i) = w_i + (0.5) \cdot \Big(t_i^3 + t_i - w_i - \frac{1}{2}(0.5)t_i^3 + \frac{3}{2}(0.5)t_i^2 - \frac{1}{2}(0.5)t_i + \frac{1}{2}(0.5) + \frac{1}{2}(0.5)w_i + \frac{1}{6}(0.5)^2t_i^3 - \frac{1}{2}(0.5)^2t_i^2 + \frac{7}{6}(0.5)^2t_i - \frac{1}{6}(0.5)^2 - \frac{1}{6}(0.5)^2w_i - \frac{1}{24}(0.5)^3t_i^3 + \frac{1}{8}(0.5)^3t_i^2 - \frac{7}{24}(0.5)^3t_i + \frac{7}{24}(0.5)^3 + \frac{1}{24}(0.5)^3w_i\Big)$$
$w_1 = 0.7291666667$

$w_2 = 0.9453667535$

$w_3 = 1.912162639$

$w_4 = 4.084398164$