Math 210
Exam 2 **Name:**______________________________

Instructions: You will have 55 minutes to complete this exam. The credit given on each problem will be proportional to the amount of correct work shown. Answers without supporting work will receive little credit.

1. (3 points each) Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{2, 5\}$, $B = \{3, 5, 7, 9\}$, and $C = \{2, 4, 5, 6, 7\}$

   (a) Find $\overline{C - A}$

   First notice that $C - A = \{4, 6, 7\}$.

   Therefore, taking the set complement relative to the set U given above, $\overline{C - A} = \{1, 2, 3, 5, 8, 9, 10\}$.

   (b) Find $A \cup (B \cap C)$

   First notice that $B \cap C = \{5, 7\}$.

   Therefore, $A \cup (B \cap C) = \{2, 5, 7\}$.

   (c) List $B \times A$ in roster notation.

   $B \times A = \{(3, 2), (3, 5), (5, 2), (5, 5), (7, 2), (7, 5), (9, 2), (9, 5)\}$

   (d) Find $|\mathcal{P}(C)|$

   As discussed in class, since $|C| = 5$, using the "inclusion/exclusion" principle, $|\mathcal{P}(C)| = 2^5 = 32$.

2. (3 points each) Let $A = \{0, \{0, 1\}\}$. True or False (You do not need to justify your answers).

   (a) $\{\emptyset\} \subset A$.

   False. ($\emptyset$ is a subset of any set, but $\{\emptyset\} \neq \emptyset$).

   (b) $\emptyset \in A$

   False. ($\emptyset$ is not a listed element of A).

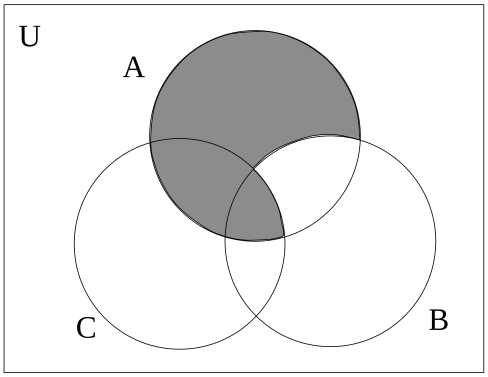
   (c) $1 \in A$

   False. (1 is not a listed element of A).

   (d) $\{0, 1\} \subseteq A$

   False. (This would require both 0 and 1 to be elements of A).

3. (5 points) Draw a Venn Diagram representing the set $A - (B - C)$



4. (7 points) Determine whether or not $A - (B \cup C) = (A - B) \cap (A - C)$

A	B	C	$B \cup C$	$A - (B \cup C)$
1	1	1	1	0
1	1	0	1	0
1	0	1	1	0
1	0	0	0	1
0	1	1	1	0
0	1	0	1	0
0	0	1	1	0
0	0	0	0	0

A	B	C	$A - B$	$A - C$	$(A - B) \cap (A - C)$
1	1	1	0	0	0
1	1	0	0	1	0
1	0	1	1	0	0
1	0	0	1	1	1
0	1	1	0	0	0
0	1	0	0	0	0
0	0	1	0	0	0
0	0	0	0	0	0

Since the last columns of these membership tables are identical, these two sets are equal.

Note: I also would have accepted a two column proof.

5. (4 points each) Write the first four terms of the following sequences. Start with $n = 1$ and be sure to fully simplify each term.

(a) $a_n = 2^n - n^2$

$a_1 = 2 - 1 = 1$, $a_2 = 2^2 - 2^2 = 0$,
$a_3 = 2^3 - 3^2 = -1$, $a_4 = 2^4 - 4^2 = 0$

(b) $b_n = \dfrac{1}{b_{n-1}} + 1; b_0 = 2$

$b_1 = \frac{1}{2} + 1 = \frac{3}{2}$, $b_2 = \frac{1}{\frac{3}{2}} + 1 = \frac{2}{3} + 1 = \frac{5}{3}$,
$b_3 = \frac{1}{\frac{5}{3}} + 1 = \frac{3}{5} + 1 = \frac{8}{5}$, $b_4 = \frac{1}{\frac{8}{5}} + 1 = \frac{5}{8} + 1 = \frac{13}{8}$

6. (4 points each) For each given sequence, find a recurrence relation that satisfies the sequence.

   (a) $a_n = 3n - 4$

   $a_n = a_{n-1} + 3$; $a_0 = -4$ (Notice that the terms of the sequence increase by 3 each time)

   (b) $b_n = n!$

   $b_n = b_{n-1} \cdot n$; $b_0 = 1$ (this comes directly form the definition of the factorial function and was discussed in class)

   (c) $c_n = 3^n$

   $c_n = 3 \cdot c_{n-1}$; $c_0 = 1$ (Notice that consecutive terms differ by a multiple of three)

7. (8 points) Show that $a_n = 4^{n-1}(2n-1)$ is a solution to the recurrence relation $a_n = 8a_{n-1} - 16a_{n-2}$

   Notice that if $a_n = n4^n$ for all n, then $a_{n-1} = 4^{n-2}(2(n-1)-1) = 4^{n-2}(2n-3)$ and $a_{n-2} = 4^{n-3}(2(n-2)-1) = 4^{n-3}(2n-5)$

   Therefore, $8a_{n-1} - 16a_{n-2} = 8(4^{n-2}(2n-3)) - 16(4^{n-3}(2n-5)) = 2(4)4^{n-2}(2n-3) - 4^2 4^{n-3}(2n-5) = 4^{n-1}(4n-6) - 4^{n-1}(2n-5) = 4^{n-1}[4n-6-(2n-5)] = 4^{n-1}[4n-6-2n+5] = 4^{n-1}(2n-1)$, so this sequence is a solution to the given recurrence relation.

8. Find the value of the following sums:

   (a) (3 points) $\sum_{k=1}^{4} \left(2^k - k^2\right)$

   $= (2-1) + (4-4) + (8-9) + (16-16) = 1 + 0 - 1 + 0 = 0$

   (b) (5 points) $\sum_{i=0}^{3} \sum_{j=1}^{4} (j^2 - i)$

   $$\sum_{i=0}^{3} ((1-i) + (4-i) + (9-i) + (16-i)) = \sum_{i=0}^{3} (30-4i)$$

   $= (30-0) + (30-4) + (30-8) - (30-12) = 30 + 26 + 22 + 18 = 96.$

9. Let $f : \mathbb{N} \to \mathbb{N}$ be defined by $f(n) = \left\lceil \frac{n}{2} \right\rceil$

(a) (7 points) Determine whether or not f is one-to-one. Justify your answer.

This function is not one-to-one. For example, $f(1) = \left\lceil \frac{1}{2} \right\rceil = 1$, while $f(2) = \left\lceil \frac{2}{2} \right\rceil = \lceil 1 \rceil = 1$, but $1 \neq 2$.

(b) (7 points) Determine whether or not f is onto. Justify your answer.

This function is onto. To see this, we consider an arbitrary $k \in \mathbb{N}$. Let $n = 2k$.
Then $f(n) = \left\lceil \frac{2k}{2} \right\rceil = \lceil k \rceil = k$.

10. (14 points) Use mathematical induction to prove the following: $\sum_{i=1}^{n} (2i - 1) = n^2$.

Base Case: $P(1)$: Notice that $\sum_{i=1}^{1} (2i - 1) = 2(1) - 1 = 2 - 1 = 1$. While $1^2 = 1$, so $P(1)$ is verified.

Inductive Step: Suppose that $P(k)$ is true. That is, assume that $\sum_{i=1}^{k} (2i - 1) = k^2$.

We must show $P(k+1)$: $\sum_{i=1}^{k+1} (2i - 1) = (k+1)^2$.

Notice that $\sum_{i=1}^{k+1} (2i - 1) = \sum_{i=1}^{k} (2i - 1) + 2(k+1) - 1$.

Applying the induction hypothesis, this is equal to $k^2 + 2(k+1) - 1 = k^2 + 2k + 2 - 1 = k^2 + 2k + 1 = (k+1)^2$.

This proves the theorem. $\square$

Extra Credit: The sequence C_n is defined as follows. C_n is the number of ways in which n coins can be arranged in horizontal rows with all the coins in each row touching and every coin above the bottom row touching two coins in the row below it.

(a) (3 points) Find C_1, C_2, C_3, C_4 and C_5

(b) (3 points) Find a recurrence relation that the terms of the sequence you found satisfy.

Come talk to me if you want to hear more about how to solve this problem.