Math 210
Exam 2 - Practice Problems

1. For each of the following, determine whether the statement is True or False.

   (a) $\emptyset \subseteq \{a, b, c, d\}$
   (b) $\emptyset \in \{a, b, c, d\}$
   (c) $\emptyset \in \{a, b, \emptyset\}$
   (d) $\emptyset \subseteq \{a, b, \emptyset\}$
   (e) $\{a, b\} \subset \{a, b\}$
   (f) $0 \in \{0, \{1\}, \{0, 1\}\}$
   (g) $1 \in \{0, \{1\}, \{0, 1\}\}$
   (h) $\{0, 1\} \in \{0, \{1\}, \{0, 1\}\}$
   (i) $\{0, 1\} \subset \{0, \{1\}, \{0, 1\}\}$

2. Given the set $B = \{a, b, \{a, b\}\}$ (a) Find $|B|$. (b) Find $\mathcal{P}(B)$

3. Given that $A = \{1, 2, 3\}$ and $B = \{a, b, c, d, e, f\}$

   (a) List the elements in $A \times A$.
   (b) How many elements are in $A \times B$?
   (c) How many elements are in $A \times (B \times B)$?

4. Find the set of all elements that make the predicate $Q(x) : x^2 < x$ true (where the domain of x is all real numbers).

5. Given that $A = \{0, 2, 4, 6, 8, 10, 12\}$, $B = \{0, 2, 3, 5, 7, 11, 12\}$ and $C = \{1, 2, 3, 4, 6, 7, 8, 9\}$ are all subsets of the universal set $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, find each of the following:

   (a) $A - B$
   (b) $\overline{A}$
   (c) $A \cap B$
   (d) $A \cup (B \cap C)$
   (e) $A - (\overline{B} \oplus C)$
   (f) $(A \cap C) \cup (B - \overline{A})$

6. Draw Venn Diagrams representing each of the following sets:

   (a) $A - B$
   (b) $B - \overline{A}$
   (c) $(A \cup C) \cap B$
   (d) $\overline{A \cup B \cup C}$
   (e) $A - (B \cup C)$
   (f) $(A \cap B) - \overline{C}$

7. Use a membership table to show that $(B - A) \cup (C - A) = (B \cup C) - A$.

8. Use a 2-column proof to verify the set identity: $A \cup (A \cap B) = A$.

9. For each of the following, either prove the statement or show that it is false using a counterexample.

   (a) $(A - B) - C = A - (B - C)$
   (b) $A \oplus (B \oplus C) = (A \oplus B) \oplus C$
   (c) $A \cap (B - C) = (A \cap B) - (A \cap C)$

10. Consider the function $f(x) = |x|$

    (a) Suppose that the domain of this function is $\mathbb{R}$ and the co-domain is $\mathbb{R}$. Find the range of f. Is f 1-1? Is f onto? Justify your answers.
    (b) Suppose that the domain of this function is $\mathbb{N}$ and the co-domain is $\mathbb{N}$. Find the range of f. Is f 1-1? Is f onto? Justify your answers.
    (c) Suppose $S = \{-2, -1, 0, 1, 2\}$. Find $f(S)$ (the image of the set S under f). Find $f^{-1}(S)$ (the preimage of the set S under f).

11. For each of the following functions, determine whether f is a one-to-one. Also determine whether f is onto. Justify your answers.

    (a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3 - x$
    (b) $f : \mathbb{R}^+ \to \mathbb{R}^+$ $f(x) = x^2$
    (c) $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ $f(m, n) = m^2 - n$
    (d) $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ $f(m, n) = m^2 - n^2$

12. Prove or Disprove: Suppose $f : B \to C$ and $g : A \to B$. If f is one-to-one and g is onto, then $f \circ g$ is one to one.

13. Prove or Disprove: Suppose $f : B \to C$ and $g : A \to B$. If f is one-to-one and g is onto, then $f \circ g$ is onto.

14. Find the first five terms of each of the following sequences (start with $n = 1$):

(a) $a_n = 3n - 1$

(b) $b_n = (-1)^n n^2$

(c) $c_n = n^{n-1}$

(d) $a_n = 2a_{n-1}$, $a_0 = 5$

(e) $b_n = b_{n-1} + n^2$, $b_0 = 7$

(f) $c_n = c_{n-1} + nc_{n-2}$, $c_0 = 2$, $c_1 = 3$

15. Find three different sequences beginning with the terms $a_1 = 1$, $a_2 = 2$ and $a_3 = 4$.

16. Determine whether or not each of the following is a solution to the recurrence relation $a_n = 8a_{n-1} - 16a_{n-2}$

(a) $a_n = 0$

(b) $a_n = 2^n$

(c) $a_n = n4^n$

(d) $a_n = 2 \cdot 4^n + 3n \cdot 4^n$

17. Find a recurrence relation satisfying each of the following:

(a) $a_n = 3n - 2$

(b) $a_n = 3^n$

(c) $a_n = n^2$

18. Find the solution to each of the following recurrence relations and initial conditions

(a) $a_n = 4a_{n-1}$, $a_0 = 1$

(b) $a_n = a_{n-1} + 4$, $a_0 = 4$

(c) $a_n = a_{n-1} + n$, $a_0 = 1$

19. Compute the value of each of the following summations:

(a) $\sum_{k=1}^{5} 2k$

(b) $\sum_{i=0}^{3} 3^i$

(c) $\sum_{j=3}^{13} 5$

(d) $\sum_{j=2}^{5} 2^j - 2j$

(e) $\sum_{i=1}^{3}\sum_{j=0}^{2} ij^2$

(f) $\sum_{j=0}^{2}\sum_{i=1}^{2} ij^2$

20. Prove that $n^5 - n$ is divisible by 5 for any non-negative integer n.

21. Prove that for $r \in \mathbb{R}$, $r \neq 1$ and for all integers n, $\sum_{j=0}^{n} r^j = \frac{r^{n+1} - 1}{r - 1}$

22. Prove that for all $n \geq 2$, $\sum_{k=1}^{n} \frac{1}{k^2} < 2 - \frac{1}{n}$

23. Prove that $n! < n^n$ whenever $n > 1$.

24. Prove that for all n, $\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} = \frac{n}{2n+1}$