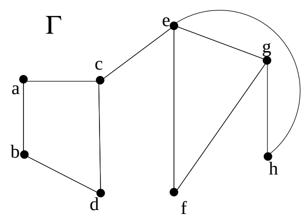
Math 210
Exam 3 **Name:**______________________________

Instructions: You will have 55 minutes to complete this exam. The credit given on each problem will be proportional to the amount of correct work shown. Answers without supporting work will receive little credit.

1. Given the following undirected graph Γ:



(a) (5 points) Find the adjacency matrix for this graph.

Using the standard alphabetic ordering on vertices, the following is the adjacency matrrix for Γ:

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

(b) (5 points) Does Γ have an Euler Circuit? What about an Euler Path? Justify your answer.

Notice that the degree sequence for this graph is: 4, 3, 3, 2, 2, 2, 2, 2.

Since there are two vertices of odd degree, Γ cannot have an Euler Circuit.

Since there are *exactly* two vertices of odd degree, Γ does have an Euler Path. Finding one is not difficult, but was not required.

One example is: $\langle c, a, b, d, c, e, f, g, h, e, g\rangle$.

(c) (5 points) Does Γ have a Hamiltonian Circuit? What about a Hamiltonian Path? Justify your answer.

Notice that the edge $\{c, e\}$ is a bridge (cut edge). For this reason, Γ does not have a Hamiltonian Circuit.

To see this, notice that if the circuit were to start at either a, b, or d, then one must eventually visit c and then traverse the edge $\{c, e\}$ to get to vertex e. It is then impossible to return to the start vertex without revisiting the vertex c.

If the circuit were to start at c, one is forced to either traverse the cycle $\langle c, a, b, d, c\rangle$, or reuse an edge in this cycle, or begin by traversing the edge $\{c, e\}$, making a, d and d inaccessible without revisiting vetex c.

A similar argument shows that one cannot contruct a Hamiltonian Circuit starting at e, f, or g.

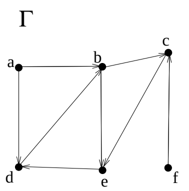
Notice that Γ does have a Hamiltonian Path (several, in fact). One example is: $\langle a, b, d, c, e, f, g, h\rangle$.

(d) (5 points) Verify that the Handshaking Theorem Holds for Γ.

First notice that $|E| = 10$. Next, notice that if we add the degree sequence for Γ, we get: $4+3+3+2+2+2+2+2 = 20$.

Therefore, $\sum_{v \in V} deg(v) = 20 = 2 \cdot |E|$, so the Handshaking Theorem does hold for this graph.

2. Based on the directed graph Γ shown below:



(a) (4 points) Find $deg^{-}(b)$

$deg^{-}(b) = 2$

(b) (4 points) Find $deg^{+}(e)$

$deg^{+}(e) = 1$

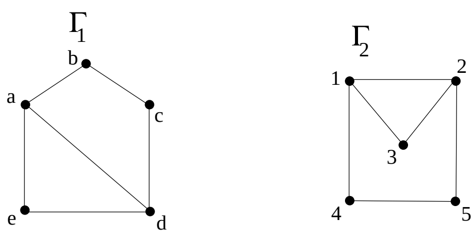
(c) (6 points) Find the strongly connected components of Γ.

Noting the directed circuit from b to c to e to d and back to b, we see that all four of these vertices are in the same strongly connected component. Since a has only outgoing edges, a is in a component all by itself. Similarly, since f has only outgoing edges, f is in a component all by itself. Hence the strongly connected components of this directed graph are:

$\{b, c, d, e\}$, $\{a\}$, and $\{f\}$. Recall that every vertex of the graph is in some strongly connected component.

3. (6 points each) For each pair, find an isomorphism from Γ_1 to Γ_2 or prove that no such isomorphism exists.

(a)



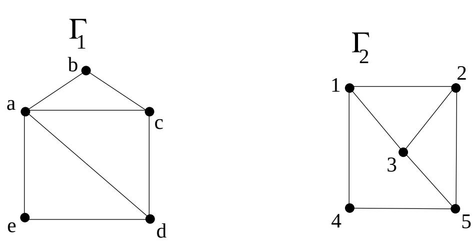
We will construct an isomorphism between Γ_1 and Γ_2.

Here is one possible isomorphism:

Γ_1 Vertex	Γ_2 Vertex	Γ_1 Edge	Γ_2 Edge
a	1	$\{a, b\}$	$\{1, 4\}$
b	4	$\{b, c\}$	$\{4, 5\}$
c	5	$\{c, d\}$	$\{2, 5\}$
d	2	$\{a, d\}$	$\{1, 2\}$
e	3	$\{a, e\}$	$\{1, 3\}$
		$\{d, e\}$	$\{2, 3\}$

Since this function is a bijection between vertices that preserves edges, Γ_1 and Γ_2 are isomorphic.

(b)



Notice that these graphs cannot be isomorphic since their degree sequences are: $4, 3, 3, 2, 2$ and $3, 3, 3, 3, 2$ respectively, and degree sequences are preserved by graph isomorphism.

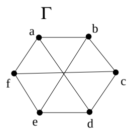
4. (6 points each) Draw a simple graph with each of the following degree sequences or state why one is not possible.

   (a) 3, 3, 3, 3, 3

   This degree sequence is not possible. Notice that the sum of the degrees in the sequence is 15. However, according to the Handshake Theorem, the sum of the degrees of the vertices of a simple graph must be even.

   (b) 3, 3, 3, 3, 3, 3

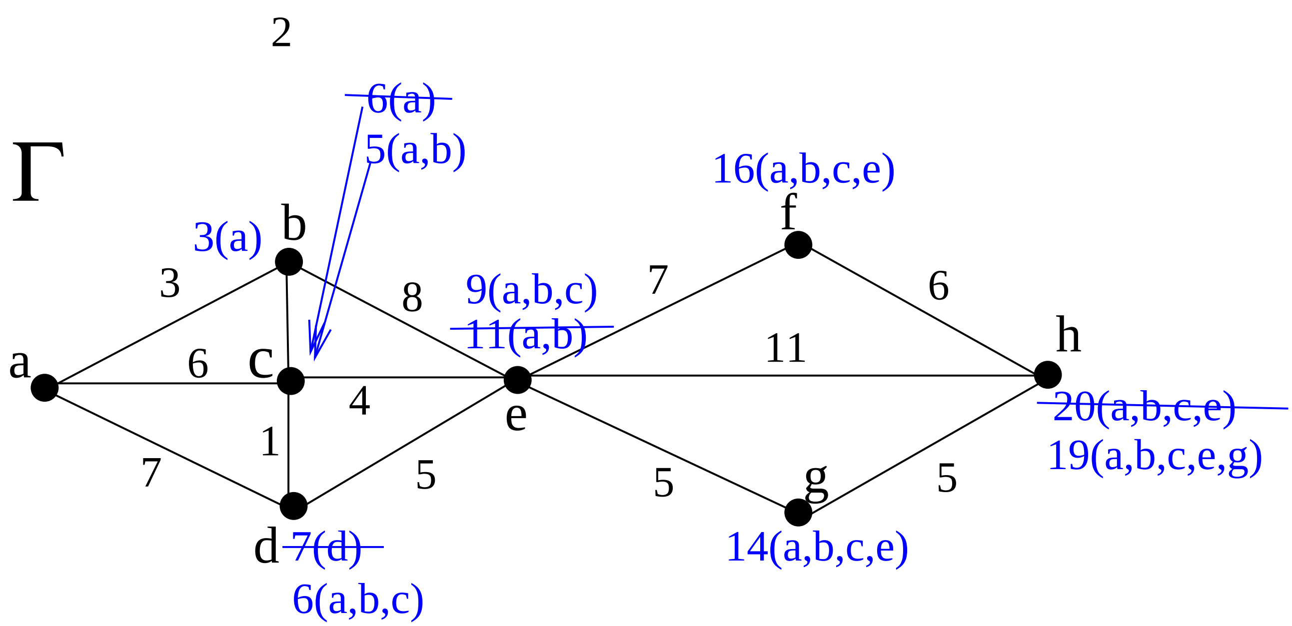
   This degree sequence is possible. Here is one example:



   (c) 3, 2, 1, 0

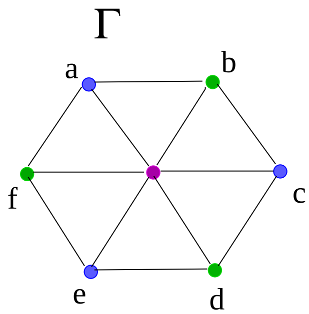
   This degree sequence is also impossible, but for a more subtle reason. Notice that there are 4 vertices listed in the sequence. Also, one vertex has degree 3. In a simple graph, neither loops nor multiedges are allowed. For this reason, a vertex of degree 3 must be adjacent to exactly 3 other distinct vertices. However, since one vertex in this sequence is an isolated vertex (the one that corresponds to the zero in the sequence) there are only 2 other distinct vertices available that could be adjacent to the vertex with degree 3. Therefore, this degree sequence is not possible for a simple graph.

5. (8 points) Use Dijkstra's Algorithm to find a shortest path from a to h in the weighted graph shown. Be sure to include all vertex labels from the algorithm.



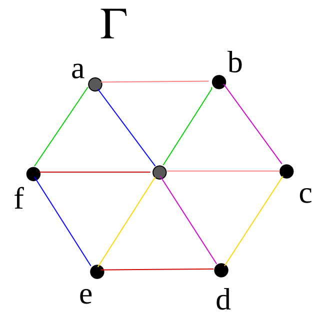
From this, we see that the shortest path from a to h is: $\langle a, b, c, e, g, h \rangle$ which has a total weight of 19.

6. (a) (6 points) Find the vertex chromatic number for W_6. Justify your answer.



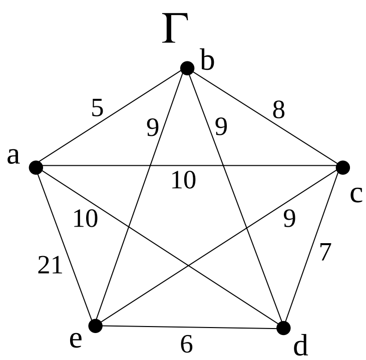
The diagram shown above demonstrates a 3-coloring of the vertices of W_6. Also notice that we cannot use less than 3 colors, since the outside ring is isomorphic to C_6, which requires 2 colors, and the central vertex is connected to every other vertex, so it must have its own color. Hence the Vertex Chromatic Number for W_6 is 3.

(b) (6 points) Find the edge chromatic number for W_6. Justify your answer.



The diagram shown above demonstrates a 6-coloring of the edges of W_6. Also notice that we cannot use less than 6 colors, since the central vertex has degree 6, and each edge incident to it must be have a different color then the others. Hence the Edge Chromatic Number for W_6 is 6.

7. Given the following weighted graph:



(a) (4 points) Use a greedy algorithm to find a solution to the traveling salesman problem for the given graph.

Recall that using a greedy algorithm requires us to select the cheapest available edge that still allows the construction of a Hamiltonian Circuit for the given graph. Carrying this out, we are forced to add the edges of the circuit around the outside of the graph, eventually forcing us to select $\{a, e\}$ as the final edge in the Hamiltonian Circuit, which has weight 21 (ouch).
This yields the final route: $\langle a, b, c, d, e, a \rangle$, which has total weight $5 + 8 + 7 + 6 + 21 = 47$.

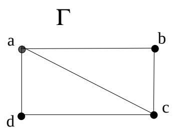
(b) (4 points) Find a different solution that is better then the first one you found.

There were many possible circuits that are better than the one found by the greedy algorithm. Any of these received full credit, as long as you found one that was a Hamiltonian Circuit with total weight less than 47. One possibility is: $\langle a, b, e, d, c, a \rangle$, which has total weight $5 + 9 + 6 + 7 + 10 = 37$.

8. (5 points) For each of the following, either produce a specific example or explain why no such graph exists. If you produce an example, no further justification us needed.

   (a) A graph that that has a Hamiltonian Circuit but no Euler Circuit.

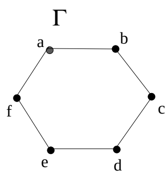
   Many possible examples exist. Here is one:



   Note that $\langle a, b, c, d, a \rangle$ is a Hamiltonian Circuit, and no Euler Circuit exists since a and c have odd degree.

   (b) A graph that does not satisfy the hypotheses of Dirac's Theorem but that has a Hamiltonian Circuit.

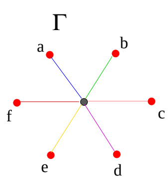
   Many possible examples exist. Here is one:



   Note that this graph has a Hamiltonian Circuit: $\langle a, b, c, d, e, f, a \rangle$. However, $n = 6$, so to satify Dirac's Theorem, we would need each vertex to have degree at least $\frac{n}{2} = 3$. Since this is a connected simple graph in which every vertex has degree 2, it does not meet the hypotheses of Dirac's Theorem.

   (c) A graph with a vertex chromatic number of 2 and an edge chromatic number of 6.

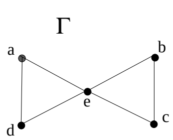
   Many possible examples exist. Here is one:



   The coloring shown in the diagram above (along with an argument similar to problem 6(b)) demonstrates that we have found a graph with the desired properties.

   (d) A graph that has a Euler Circuit but no Hamiltonian Circuit.

   Many possible examples exist. Here is one:



   Note that every vertex has even degree, so this graph does have an Euler Circuit. Any attempt to construct a Hamiltonian Circuit will require is to visit the vertex e twice before completing the curcuit, so this graph does not have a Hamiltonian Circuit.