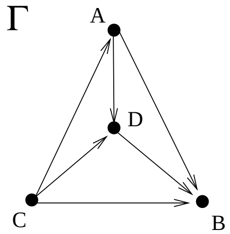
Math 310
Exam 3 - Practice Problem Solutions

1. Suppose the following graph represents the result of a round robin volleyball tournament. Give the won lost record for each team who participated. Who won the tournament?



Recall that the in-degree of a vertex in a directed graph representing a round robin tournament gives the number of losses for that team.

Similarly, the out-degree of a vertex in a directed graph representing a round robin tournament gives the number of wins for that team.

Using this, Team A has a record of $(2-1)$, Team B has a record of $(0-3)$, Team C has a record of $(3-0)$, and Team D has a record of $(1-2)$.

Therefore, Team C won the tournament.

2. State what type of graph model you would use for each of the following. Make it clear what the vertices in your model represent, what the edges represent, whether the edges are directed or undirected, and whether not not loops and multiedges are allowed.

   (a) A graph model for the streets in Moorhead, Minnesota.

   We will take vertices to be intersections and edges to be roads (or lanes of roads) between intersections.

   Since Moorhead has some one-way streets, we need a directed graph. Loops do not make sense in this context. We could choose to use multiedges of we wish to account for the streets that have more than one lane heading in a particular direction.

   (b) A graph that models the molecular structure of a certain protein molecule.

   We will take vertices to be atoms and edges to be chemical bonds between atoms.

   Chemical bonds are generally thought of as being between a pair of atoms rather than from one atom to another, so one would normally use an undirected graph.

   We generally do not think of bonds from an atom to itself, so there would be no loops.

   Since there "multiple bonds" in some molecules, we would make use of multi-edges.

   (c) A graph modeling all the phone calls made on a certain carrier during the last hour.

   If we take our vertices to be phone numbers, and edges to be calls originating from one number and connecting with another number, then we clearly need directed edges. We could deal with conference calls by drawing an edge from the person originating the conference call to each of the other individual lines involved in the call.

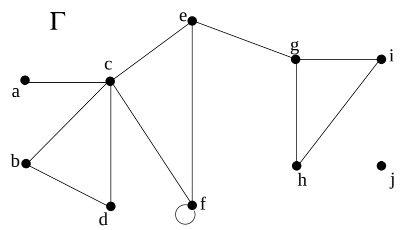
   Loops do not make sense in this context, since a person cannot call himself (at least not at the same number). We would allow multi-edges, since a peron could call the same person more than once during a single hour.

   (d) A graph modeling a computer network.

   Here, loops do not make sense. Whether one uses multiedges and/or multi-edges depends on how precisely you think about how connections are made between computers (server/slave relationships and/or multiple ports could be taken into account).

3. Given the following undirected graph:

Γ



(a) Find $|E|$
$|E| = 12$

(b) Find $|V|$
$|V| = 10$

(c) Find $deg(c)$, $deg(f)$, and $deg(g)$
$deg(c) = 5$, $deg(f) = 4$, and $deg(g) = 3$

(d) List all isolated vertices of Γ
j is an isolated vertex.

(e) List all pendant vertices of Γ
a is a pendant vertex.

(f) Is Γ connected? Justify your answer. How many connected components does Γ have?
Γ is not connexted, since it has an isolated vertex. Γ has two connecte components since every pair of vertices not involving j is connected.

(g) List all bridge edges in Γ.
Γ has two bridge edges: $\{a, c\}$ and $\{e, g\}$

(h) List all cut vertices in Γ.
c, e, and g are cut vertices.

(i) Find the set of vertices adjacent to vertex c.
$\{a, b, d, e, f\}$.

(j) Find the degree sequence for Γ.
$5, 4, 3, 3, 2, 2, 2, 2, 1, 0$

(k) Verify that the Handshaking Theorem holds for Γ.
According to the Handshaking Theorem, $\sum_{v \in V} deg(v) = 2|E|$.

Here, $\sum_{v \in V} deg(v) = 5 + 4 + 3 + 3 + 2 + 2 + 2 + 2 + 1 + 0 = 24 = 2(12) = 2|E|$. Therefore, the Handshaking Theorem is verified for this graph.

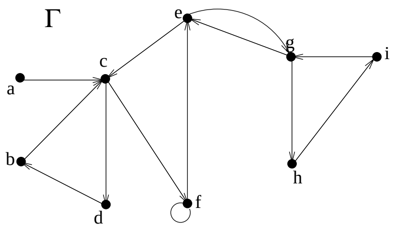
(l) Form the adjacency matrix for Γ with the vertices ordered alphabetically.

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(m) Form an incidence matrix for Γ with the vertices ordered alphabetically and the edges in an order of your choosing.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4. Given the following directed graph:



(a) Find $|E|$
$|E| = 13$

(b) Find $|V|$
$|V| = 9$

(c) Find $deg^-(c)$, $deg^-(f)$, $deg^+(c)$, and $deg^+(f)$.
$deg^-(c) = 3$, $deg^-(f) = 2$, $deg^+(c) = 2$, and $deg^+(f) = 2$.

(d) Is Γ strongly connected? Justify your answer. How many strongly connected components does Γ have?
Γ is not strongly connected because there are no paths into vertex a.
Γ has two strongly connected components: $\{a\}$ and $\{b, c, d, e, f, g, h, i\}$

(e) Γ weakly connected? Justify your answer.
Yes. The underlying graph is connected, so Γ is weakly connected.

(f) Find the set of vertices adjacent to vertex c.
The vertices $\{a, b, e\}$ are adjacent to vertex c.

(g) Find the set of vertices adjacent from vertex c.
The vertices $\{d, f\}$ are adjacent from vertex c.

(h) Form the adjacency matrix for Γ with the vertices ordered alphabetically.

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

5. Use the Handshaking Theorem to prove that an undirected graph has an even number of vertices of odd degree.

Proof:

According to the Handshake Theorem, for any undirected graph Γ, $\sum_{v \in V} deg(v) = 2|E|$. That is, the sum of the elements of the degree sequence of a graph is even.

Let V_1 be the set of vertices of Γ with even degree, and let V_2 be the set of vertices of Γ of add degree. Let $k = \sum_{v \in V_1} deg(v)$.

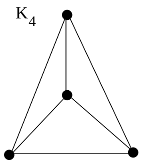
Since the sum two even numbers is even, k is even.

Let $m = \sum_{v \in V_2} deg(v)$ Then, $2|E| - k = m$ is also even, since the difference of two even numbers is even. If we suppose that $|V_2|$ is odd, then $m = \sum_{v \in V_2} deg(v)$ is odd, since the sum of an odd number of odd integers is odd. This is a contradiction, since m has already been shown to be even.

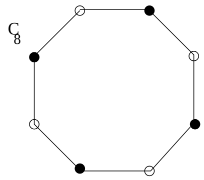
Hence $|V_2|$ must be even. That is, Γ must have an even number of vertices with odd degree.

6. (a) Draw each of the following graphs:

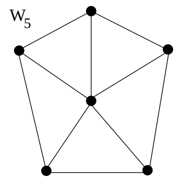
i. K_4



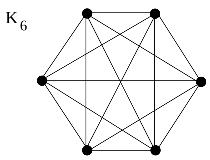
ii. C_8



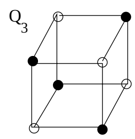
iii. W_5



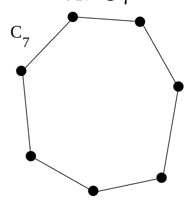
iv. K_6



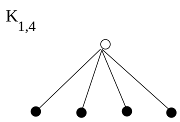
v. Q_3



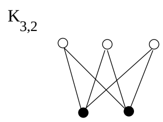
vi. C_7



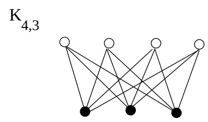
vii. $K_{1,4}$



viii. $K_{3,2}$



ix. $K_{4,3}$

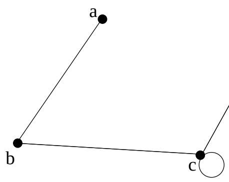


(b) Which of the graphs above are bipartite? Justify your answer.

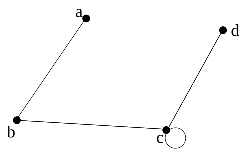
ii, v, vii, viii, and ix are bipartite. A 2-coloring has been included in the graphs shown above. All the others contain at least one od cycle, so that cannot be bipartite.

7. Draw all subgraphs of the following graph.

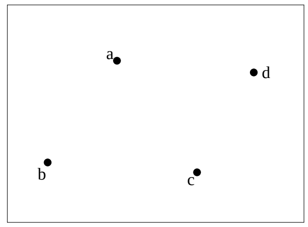
Γ

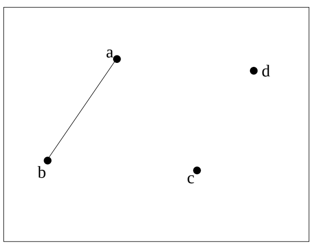


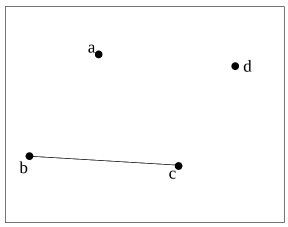
Γ

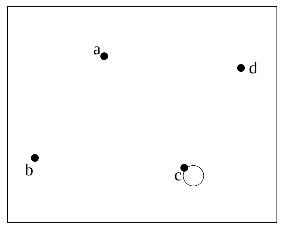


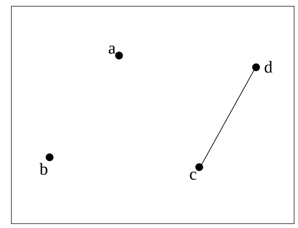
Subgraphs:

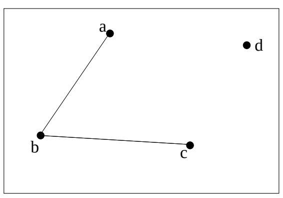


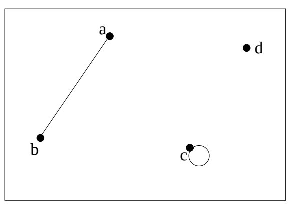


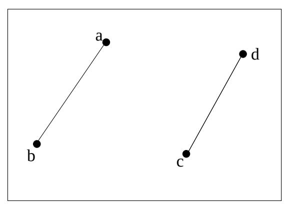


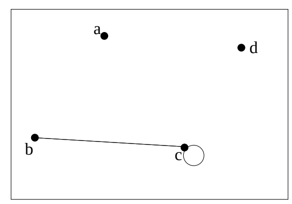


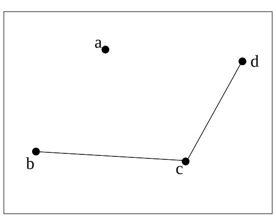


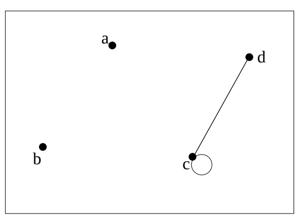


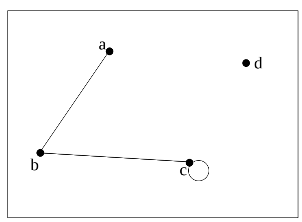


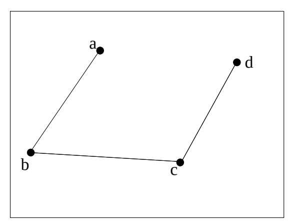


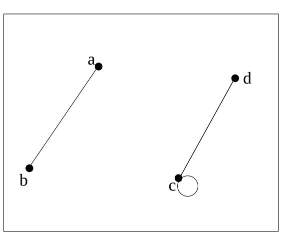


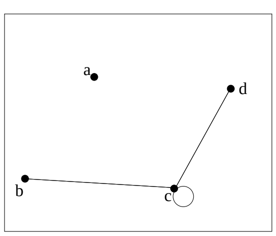


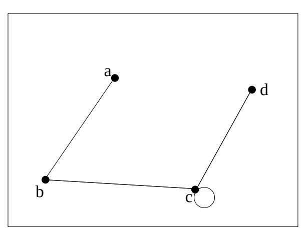




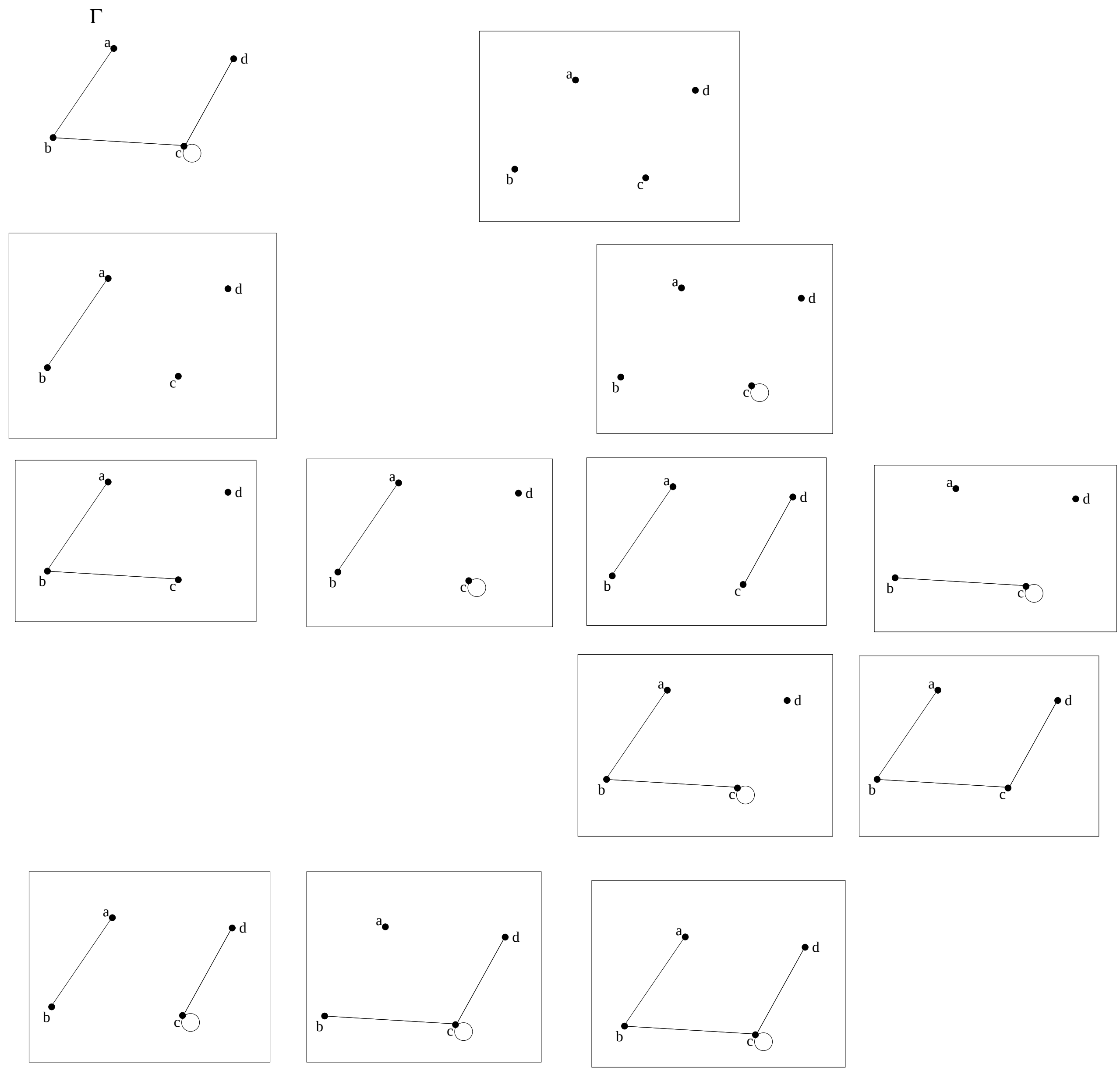




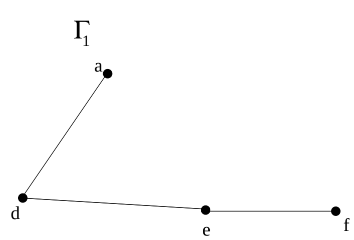


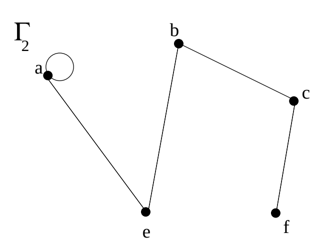


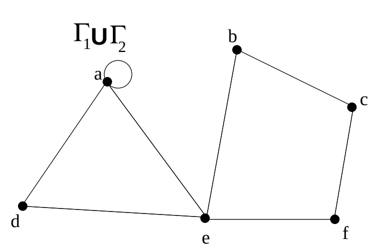
8. From the list of subgraphs you found above, draw one representative of each isomorphism class.



9. Find the union of the following graphs:

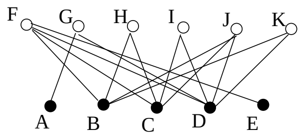




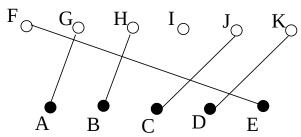


10. 5 boys (Abe, Ben, Chuck, Donald, Elmer) and 6 girls (Francine, Gretchen, Heather, Irene, Jennifer, and Katie) are trying to find dates to the junior prom. Francine is willing to go with Ben, Chuck, Donald, or Elmer. Gretchen is willing to go with Abe or Donald. Heather is willing to go with Ben or Chuck. Irene is willing to go with Chuck or Donald. Jennifer is willing to go with Ben, Chuck, or Donald. Katie is willing to go with Donald or Ben.

(a) Draw a graph modeling this situation.



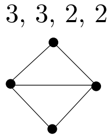
(b) Find a matching for which every boy has a date to the prom.



This matching corresponds to Abe going with Gretchen, Ben going with Heather, Chuck going with Jennifer, Donald going with Katie, and Elmer going with Francine.

11. Draw a simple graph with each of the following degree sequences or state why one is not possible.

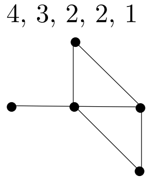
(a) 3, 3, 2, 2



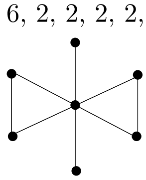
(b) 4, 3, 2, 1

Not possible. Since there are only 4 vertices total in this vertex sequence, there are only three other edges that any vertex can be adjacent to, since we are insisting on a simple graph. Therefore, we cannot have a vertex with degree 4. In fact, any vertex in a simple graph with n vertices has degree at most $n-1$.

(c) 4, 3, 2, 2, 1



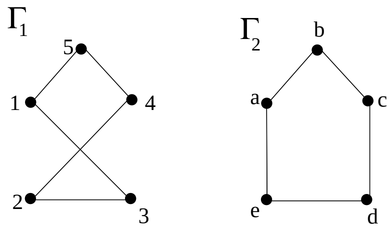
(d) 6, 2, 2, 2, 2, 1, 1



(e) 6, 2, 2, 1, 1

Not possible. As above, since there are only 5 vertices in this degree sequence, there cannot have a vertex with degree 6. In fact, any vertex in a simple graph with n vertices has degree at most $n-1$.

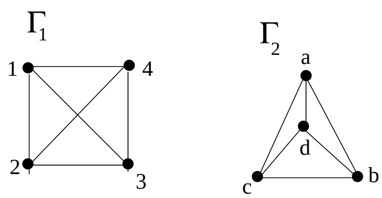
12. For each pair of graphs, prove that the graphs are isomorphic, or prove that they cannot be isomorphic.

(a)


These graphs are isomorphic. To see this, consider the following bijection between the vertex sets:

Vertex in Γ_1	Vertex in Γ_2
1	a
2	d
3	e
4	c
5	b

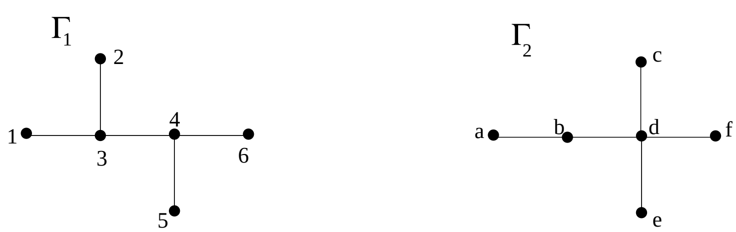
It is easy to verify that this bijection preserves adjacency, since it preserves 5-cycle present in both graphs.

(b)


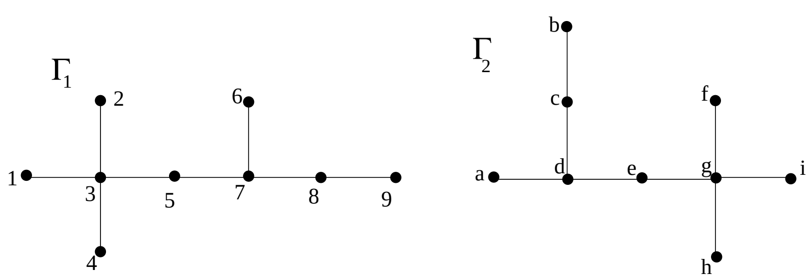
These graphs are isomorphic. To see this, consider the following bijection between the vertex sets:

Vertex in Γ_1	Vertex in Γ_2
1	a
2	c
3	d
4	b

In fact, since every pair of distinct vertices are adjacent in these graphs, any bijection between the vertices is an isomorphism [so there are $n! = 4! = 24$ diferent bijetions between these graphs].

(c)


These graphs are not isomorphic. To see this, notice that Γ_2 has degree 4, while no vertex in Γ_1 has degree 4.

(d)


These graphs are isomorphic. To see this, consider the following bijection between the vertex sets:

Vertex in Γ_1	Vertex in Γ_2
1	i
2	f
3	g
4	h
5	e
6	a
7	d
8	c
9	b

It is easy to verify that this bijection preserves adjacency, if you can visualize the tranformation by imagining straightening out the right angle in Γ_2 formed by $\angle cde$ and then rotating the graph 180^{circ}.

13. For each of the following graphs determine:

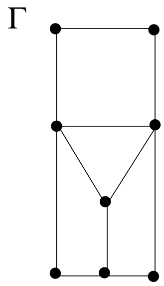
    (i) Whether or not the graph has an Euler Circuit.

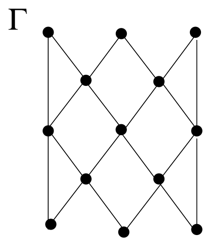
    (ii) Whether or not the graph has an Euler Path.

    (iii) Whether or not the graph has a Hamilton Circuit.

    (iv) Whether or not the graph has a Hamilton Path.

    Be sure to justify your answers.

(a) 

(b) 

Graph (a) has 2 vertices of odd degree, so it does not have an Euler Circuit, but it does have an Euler Path.

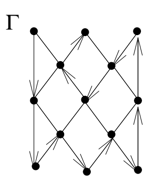
Graph (a) does not have a Hamilton Circuit. To see this, notice that both degree 4 vertices are cut vertices, but once these edges are removed, the only other edge connected to the center vertex is a bridge edge, so one cannot form a Hamilton circuit.

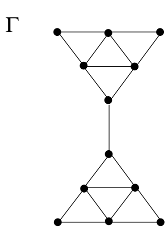
Graph (a) does have a Hamilton Path (start in the lower left corner vertex, travel clockwise though the other 6 vertices on the perimeter of the graph, and then go up to the last vertex).

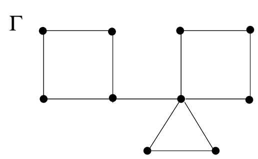
Graph (b) has vertices with even degrees, so it does have an Euler Circuit.

Graph (b) does not have a Hamilton Circuit. To see this is a bit subtle, but visiting all of the vertices in one of the main diagonal disconnects the graph so that a circuit cannot be completed.

Graph (b) does have a Hamilton Path, as exhibited by the following:



(c) 

(d) 

Graph (c) has 2 vertices of odd degree, so it does not have an Euler Circuit, but it does have an Euler Path.

Graph (c) does not have a Hamilton Circuit. To see this, notice the bridge edge in the graph.

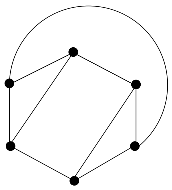
Graph (c) does have a Hamilton Path (alternate traveling left and then right through each "level" of the graph).

Graph (d) has 2 vertices of odd degree, so it does not have an Euler Circuit, but it does have an Euler Path.

Graph (c) does not have a Hamilton Circuit. To see this, notice the bridge edge in the graph.

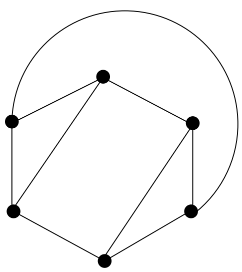
Graph (c) does note have a Hamilton Path. To see this, notice that the vertex of degree 5 is a cut vertex whose removal separates the graph in to 3 components, only two of which can be visited by and path that does not repeat this vertex.

14. Draw a graph that satisfies the hypotheses of Dirac's Theorem. Explain how you know that each hypothesis is satisfied.



Notice that this graph has 6 vertices (so more than 3). To satisfy Dirac's Theorem, each vertex needs to have degree at least $\frac{6}{2} = 3$, and we can see that each vertex has degree 3. [Note: If we think of beginning with just these six vertices and adding edges, you should be able to first convince yourself that you cannot make all vertices have degree 3 without the graph being connected. Next, convince yourself that there must be a Hamilton Circuit.]

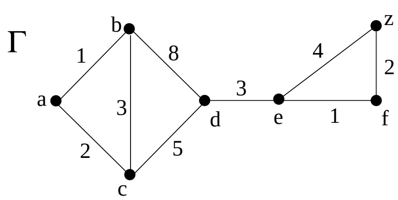
15. Draw a graph that satisfies the hypotheses of Ore's Theorem. Explain how you know that each hypothesis is satisfied. (your example may not be a complete graph.)

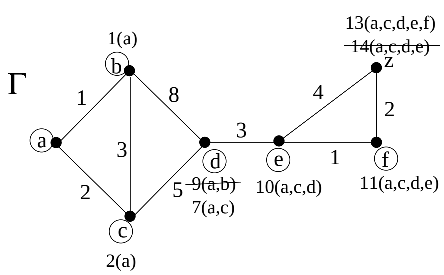


Notice that the same graph also satifies Ore's Theorem. The graph has 6 vertices (so more than 3). To satisfy Dirac's Theorem, each nonadjacent pair of vertices needs to have degrees that add up to at least $n = 6$, and we can see that since each vertex has degree 3, the sum of the degrees of any pair of nonadjacent vertices is 6.

16. For each of the following weighted graphs, use Dijkstra's Algorithm to find a shortest path from a to z.

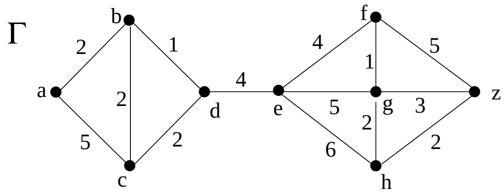
(a)

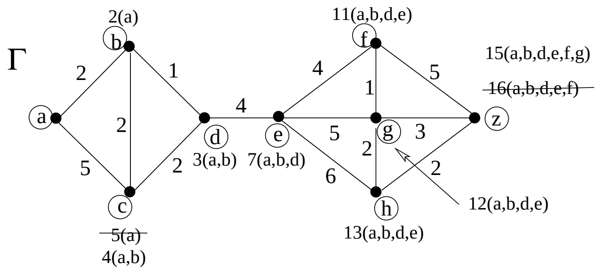




The figure above shows the result of carrying out Dijkstra's Algorithm on the given weighted graph. From this, we see that the shortest path has weight: 13 and follows the vertex sequence $\langle a, c, d, e, f, z \rangle$.

(b)

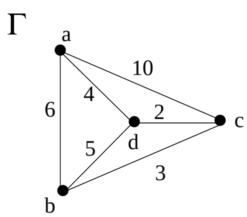




The figure above shows the result of carrying out Dijkstra's Algorithm on the given weighted graph. From this, we see that the shortest path has weight: 15 and follows the vertex sequence $\langle a, b, d, e, f, g, z\rangle$. Notice that $\langle a, b, d, e, h, z\rangle$ is also a shortest path, but it is not the path found by the algorithm.

17. Use a brute force algorithm to find an optimal solution for the traveling salesman problem for the given graph starting at "home" vertex a:

(a)

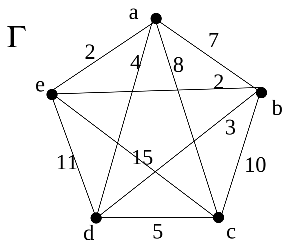


Possible Paths	Total Weight
a, b, c, d, a	$6 + 3 + 2 + 4 = 15$
a, b, d, c, a	$6 + 5 + 2 + 10 = 23$
a, c, b, d, a	$10 + 3 + 5 + 4 = 22$
a, c, d, b, a	$10 + 2 + 5 + 6 = 23$
a, d, b, c, a	$4 + 5 + 3 + 10 = 22$
a, d, c, b, a	$4 + 2 + 3 + 6 = 15$

Note that the last three paths are just the first three paths traveled in reverse, so we could have stopped after checking the first three paths.

The minimal circuits starting at a are a, b, c, d, a and a, d, c, b, a which both have total weight 15.

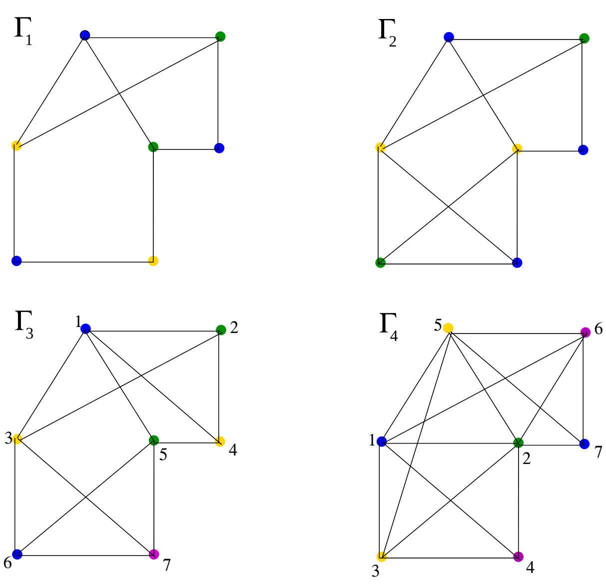
(b)



Possible Paths	Total Weight
a, b, c, d, e, a	$7 + 10 + 5 + 11 + 2 = 35$
a, b, c, e, d, a	$7 + 10 + 15 + 11 + 4 = 47$
a, b, d, c, e, a	$7 + 3 + 5 + 15 + 2 = 32$
a, b, d, e, c, a	$7 + 3 + 11 + 15 + 8 = 44$
a, b, e, c, d, a	$7 + 2 + 15 + 5 + 4 = 33$
a, b, e, d, c, a	$7 + 2 + 11 + 5 + 8 = 33$
a, c, b, d, e, a	$8 + 10 + 3 + 11 + 2 = 34$
a, c, b, e, d, a	$8 + 10 + 2 + 11 + 4 = 35$
a, d, b, c, e, a	$4 + 3 + 10 + 15 + 2 = 34$
a, d, b, e, c, a	$4 + 3 + 2 + 15 + 8 = 32$
a, e, b, c, d, a	$2 + 2 + 10 + 5 + 4 = 23$
a, e, b, d, c, a	$2 + 2 + 3 + 5 + 8 = 20$

Notice that there are 12 more paths we could check, but they are just reverses of the 12 paths that we already checked. Therefore, we know that the minimum weight circuits starting at a are: a, e, b, d, c, a and a, c, d, b, e, a, both having weight 20.

18. Find the vertex chromatic number for each of the graphs given below. You do not need to prove your answer, but you should exhibit a coloring of the vertices of the graph that verifies your answer.



The graph Γ_1 is 3-colorable. To see this, notice that since Γ_1 contains a triangle, it cannot be bipartite, so $\chi(\Gamma_1) > 2$. The coloring exhibited in the diagram above shows that $\chi(\Gamma_1) \leq 3$, so we conclude that $\chi(\Gamma_1) = 3$.

The graph Γ_2 is also 3-colorable. Notice that since Γ_2 contains a triangle, it cannot be bipartite, so $\chi(\Gamma_2) > 2$. The coloring exhibited in the diagram above shows that $\chi(\Gamma_2) \leq 3$, so we conclude that $\chi(\Gamma_2) = 3$.

The graph Γ_3 is 4-colorable. To prove this, we will constuct a 4-coloring of Γ_3 and show from the construction of this coloring that no coloring with fewer colors in possible.

As shown in the diagram above, we begin by coloring the vertices $1, 2, 3$ which form a triangle using blue, green, and gold respectively. Since vertex 4 is adjacent to both a blue vertex and a green vertex, we must color it gold is we wish to use three colors. Similarly, we then must color vertex 5 green and vertex 6 blue. This leaves us with vertex 7, which is adjacent to vertices that are blue, gold, and green, forcing us to introduce a fourth color, purple, to complete the coloring.

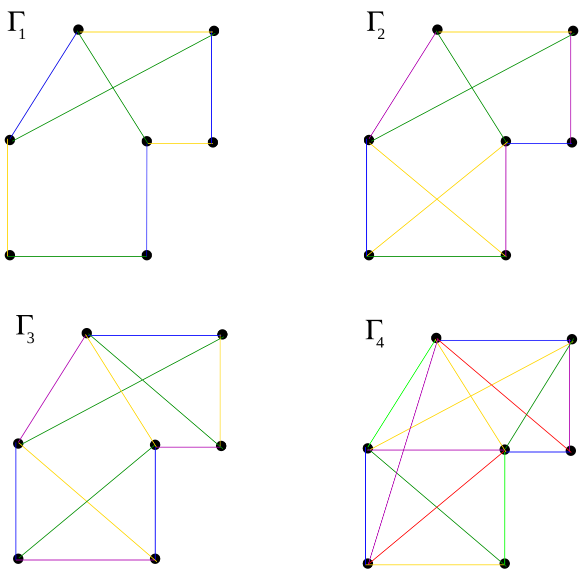
This coloring and the discussion above shows that $\chi(\Gamma_3) = 4$.

The graph Γ_4 is 4-colorable. To prove this, we will constuct a 4-coloring of Γ_4 and show from the construction of this coloring that no coloring with fewer colors in possible.

As shown in the diagram above, we begin by coloring the vertices $1, 2, 3, 4$ which form a copy of K_4 using blue, green, gold, and purple respectively. The existence of this subgraph shows that we cannot get by with fewer than 4 colors. We then continue by coloring vertex 5 gold, vertex 6 purple, and vertex 7 blue, as shown in the diagram above.

This coloring and the discussion above shows that $\chi(\Gamma_4) = 4$.

19. Find the edge chromatic number for the graphs in the previous problem.



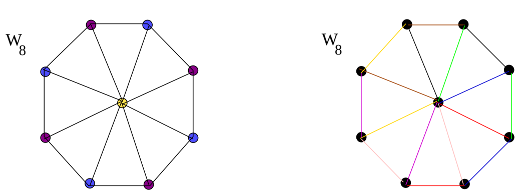
The graph Γ_1 satisfies $\chi_E(\Gamma_1) = 3$. To see this, notice that since Γ_1 has a vertex of degree 3, $\chi_E(\Gamma_1) \geq 3$. The coloring exhibited in the diagram above shows that $\chi_E(\Gamma_1) \leq 3$, so we conclude that $\chi_E(\Gamma_1) = 3$.

The graph Γ_2 satisfies $\chi_E(\Gamma_2) = 4$. To see this, notice that since Γ_2 has a vertex of degree 4, $\chi_E(\Gamma_2) \geq 4$. The coloring exhibited in the diagram above shows that $\chi_E(\Gamma_2) \leq 4$, so we conclude that $\chi_E(\Gamma_2) = 4$.

The graph Γ_3 also satisfies $\chi_E(\Gamma_3) = 4$. To see this, notice that since Γ_3 has a vertex of degree 4, $\chi_E(\Gamma_2) \geq 4$. The coloring exhibited in the diagram above is a little tricky to find, but it shows that $\chi_E(\Gamma_3) \leq 4$, so we conclude that $\chi_E(\Gamma_3) = 4$.

The graph Γ_4 satisfies $\chi_E(\Gamma_4) = 6$. To see this, notice that since Γ_4 has a vertex of degree 6, $\chi_E(\Gamma_4) \geq 6$. The coloring exhibited in the diagram above shows that $\chi_E(\Gamma_4) \leq 6$, so we conclude that $\chi_E(\Gamma_4) = 6$.

20. Let $\Gamma = W_8$. Find the vertex chromatic number and the edge chromatic number of Γ. Provide a proof for your answer.



Notice that since W_8 has a 3-cycle, it cannot be bipartite. This figure above demonstrates a 3-coloring of W_8. Hence $\chi(W_8) = 3$.

Also, since the spoke vertex of W_8 has degree 8, the edge chromatic number of W_8 is at least 8. The figure above demonstrates an 8-coloring of W_8. Thus $\chi_E(W_8) = 8$.