Math 311
Exam 1 - Fall 2015 **Name:**______________________________

**Instructions:** You will have 55 minutes to complete this exam. The credit given on each problem will be proportional to the amount of correct work shown. Answers without supporting work will receive little credit.

1. (6 points) Determine, by translating symbolically and then identifying its form, whether or not the following argument is valid:

   If I do not save a little money every week then I will not be able to go to Florida for Spring Break. I was not able to go to Florida for Spring Break. Therefore I did not save a little money every week.

   My translation will use the variables: $F$: I go to Florida for Spring Break. $S$: I save a little money every week.

   Then we have the following argument form:

   $\neg S \to \neg F$
   $\neg F$
   ---
   $\therefore \neg S$

   We see that this argument is the Fallacy of the Converse (also known as the Fallacy of Affirming the Conclusion), hence this argument is Invalid.

2. (15 points) Use a 2-column proof to show that the following argument is valid. Begin by translating each statement symbolically. Consider using the predicates $B(x)$, $I(x)$, $M(x)$ and $U(x)$.

   Everyone who is brave and intelligent majors in mathematics.
   Everyone who majors in mathematics understands logic.
   John is intelligent but does not understand logic.
   ---
   Therefore John is not brave.

   We begin by translating this argument into symbolic form. Let the domain of $x$ be the set of all people. We then defining the following predicates:

   $B(x)$ : Person $x$ is brave. $I(x)$ : Person $x$ is intelligent. $M(x)$ : Person $x$ is a math major. $U(x)$ : Person $x$ understands logic.

   Then we have the following argument:

   $\forall x\,[(B(x) \wedge I(x)) \to M(x)]$
   $\forall x\,[M(x) \to U(x)]$
   $I(\text{John}) \wedge \neg U(\text{John})$
   ---
   $\therefore \neg B(\text{John})$

   Then we can construct the following proof:

| **Statement** | **Reason** |
|---|---|
| (1) $\forall x\,[(B(x) \wedge I(x)) \to M(x)]$ | Premise |
| (2) $(B(\text{John}) \wedge I(\text{John})) \to M(\text{John})$ | Universal Instantiation of (1) |
| (3) $\forall x\,[M(x) \to U(x)]$ | Premise |
| (4) $[M(\text{John}) \to U(\text{John})]$ | Universal Instantiation of (3) |
| (5) $(B(\text{John}) \wedge I(\text{John})) \to L(\text{John})$ | Hypothetical Syllogism of (2) and (4) |
| (6) $I(\text{John}) \wedge \neg U(\text{John})$ | Premise |
| (7) $\neg U(\text{John})$ | Simplification of (6) |
| (8) $I(\text{John})$ | Simplification of (6) |
| (9) $\neg\,(B(\text{John}) \wedge I(\text{John}))$ | Modus Tollens applied to (5) and (7) |
| (10) $\neg B(\text{John}) \vee \neg I(\text{John})$ | DeMorgan's Law applied to (9) |
| (11) $\neg B(\text{John})$ | Disjunctive Syllogism (8) and (10) |

3. (15 points) Let $n$ be an integer. Prove that $n$ is even if and only if $n^2 - 1$ is odd.

Since this is an if and only if statement, there are two directions to prove.

"$\Rightarrow$": We must prove that if $n$ is even, then $n^2 - 1$ is odd.

**Proof:** We will use a direct proof. Suppose $n$ is even. Then $n = 2k$ for some $k \in \mathbb{Z}$.
Notice that $n^2 - 1 = (2k)^2 - 1 = 4k^2 - 1 = (4k^2 - 2) + 1 = 2(2k^2 - 1) + 1$. Since $k$ is an integer, $2k^2 - 1$ is also an integer. Say $2k^2 - 1 = \ell$. Then $n^2 = 2\ell + 1$. Hence $n^2 - 1$ is odd.

"$\Leftarrow$": We must prove that if $n^2 - 1$ is odd then $n$ is even.

**Proof:** We will use proof by contraposition. Suppose $n$ is odd. Then $n = 2k + 1$ for some $k \in \mathbb{Z}$.
Notice that $n^2 - 1 = (2k + 1)^2 - 1 = 4k^2 + 4k + 1 - 1 = 4k^2 + 4k = 2(2k^2 + 2k)$. Since $k$ is an integer, $2k^2 + 2k$ is also an integer. Say $2k^2 + 2k = p$. Then $n^2 = 2p$. Hence $n^2 - 1$ is even. $\square$.

4. (15 points) Use a general element argument to prove that $(A \cup B) - C \subseteq A \cup (B - C)$.

**Proof:** Let $x \in (A \cup B) - C$. Then, using the definition of set subtraction, $x \in A \cup B$ and $x \notin C$. Therefore, using the definition of set union, $x \in A$ or $x \in B$, and, in either case, $x \notin C$.

**Case 1:** Suppose $x \in A$ and $x \notin C$. Since $x \in A$, then $x \in A \cup (B - C)$.

**Case 2:** Suppose $x \in B$ and $x \notin C$. Then $x \in B - C$, hence $x \in A \cup (B - C)$.

In either case, we have $x \in A \cup (B - C)$, thus $(A \cup B) - C \subseteq A \cup (B - C)$. $\square$.

5. (15 points) Find all integer solutions to the equation $k^2 + m^2 + n^2 = 10$.

First, notice that we must have $|k| \leq 3$, $|m| \leq 3$, and $|n| \leq 3$ since whenever $|x| \geq 4$, we have $x^2 \geq 16$, which would cause our sum to exceed 10. Then $k$, $m$, and $n$ can only be one of $A = \{-3, -2, -1, 0, 1, 2, 3\}$, and their squares can only be $S = \{0, 1, 4, 9\}$. Carefully examining combinations of elements drawn from $S$, we see that the only combination that works is $0 + 1 + 9$ in some order − WLOG, we only examine cases where $k^2 \leq m^2 \leq n^2$.

$0+0+0 = 0$, $0+0+1 = 1$, $0+1+1 = 2$, $1+1+1 = 3$, $0+0+4 = 4$, $0+1+4 = 5$, $1+1+4 = 6$, $0+4+4 = 8$, $1+4+4 = 9$, $0+0+9 = 9$, $0+1+9 = 10$, and all other cases yield a result that is greater than 10.

From this, we consider both positive and negative values for the base variables $k, m$ and $n$ and allow any ordering of the variables that yield 0, 1, and 9 when squared to obtain the following complete list of solutions, which we write in the form of ordered triples: $(k, m, n)$:

$(0, 1, 3); (0, -1, 3); (0, 1, -3); (0, -1, -3)$, $(0, 3, 1); (0, -3, 1); (0, 3, -1); (0, -3, -1)$, $(1, 0, 3); (-1, 0, 3); (1, 0, -3); (-1, 0, -3)$, $(1, 3, 0); (-1, 3, 0); (1, -3, 0); (-1, -3, 0)$, $(3, 0, 1); (-3, 0, 1); (3, 0, -1); (-3, 0, -1)$, $(3, 1, 0); (-3, 1, 0); (3, -1, 0); (-3, -1, 0)$

6. (15 points) Prove that $\sqrt[3]{2}$ is irrational.

**Proof:** In order to obtain a contradiction, suppose that $\sqrt[3]{2}$ is rational. Then, by definition, $\sqrt[3]{2} = \frac{a}{b}$ for $a, b \in \mathbb{Z}$ with $b \neq 0$. Suppose that the fraction $\frac{a}{b}$ has been fully simplified. Since $\sqrt[3]{2} = \frac{a}{b}$, then $2 = \frac{a^3}{b^3}$, or, since $b \neq 0$, $2b^3 = a^3$. Therefore $a^3$ is even.

**Lemma:** If $n^3$ is even, then $n$ is even.

We will prove this lemma via contraposition. Suppose $n$ is odd. then $n = 2k+1$ for some $k \in \mathbb{Z}$. Then $n^3 = (2k+1)^3 = (4k^2+4k+1)(2k+1) = 8k^3+12k^2+6k+1 = 2(4k^3+6k^2+3k)+1$. Notice that since $k$ is an integer, $p = 4k^3+6k^2+3k$ is also an integer. Thus $n^3 = 2p+1$. Hence $n^3$ is odd. This proves the lemma.

Applying this lemma to $a^3$, since $a^3$ is even, $a$ is also even. Therefore, $a = 2\ell$ for some integer $\ell$. Then $2b^3 = a^3 = (2\ell)^3 = 8\ell^3$. From this, we see that $b^3 = 4\ell^3$. Thus $b^3$ is even, so, applying the lemma to $b^3$, $b$ is even. Since $a$ and $b$ are both even, they have a common factor of 2. This contradicts the fact that the fraction $\frac{a}{b}$ is fully simplified. Hence $\sqrt[3]{2}$ is irrational. $\square$.

7. (15 points) Let $x$ and $y$ be real numbers with $x < y$. Prove that there is a unique real number $z$ so that $x < z < y$ and $z - x = y - z$.

   **Existence:** Let $z = \frac{x+y}{2}$. Then $z - x = \frac{x+y}{2} - x = \frac{x+y}{2} - \frac{2x}{2} = \frac{-x+y}{2}$. Similarly, $y - z = y - \frac{x+y}{2} = \frac{2y}{2} - \frac{x+y}{2} = \frac{y-x}{2} = \frac{-x+y}{2}$. Moreover, since $x < y$, $\frac{x+y}{2} < \frac{y+y}{2} = y$, and $\frac{x+y}{2} > \frac{x+x}{2} = x$. Hence $x < z < y$.

   **Uniqueness:** To see that $z$ is unique, suppose $w - x = y - w$ for some real number $w$. Then, rearranging, we have $2w = x + y$, or $w = \frac{x+y}{2}$. This shows that the $z$ given above is the only real number that satisfies the given condition, hence $z$ is unique.

8. Prove **one** of the following two: (If you attempt both, make it clear which one you want to be graded).

   (a) (10 points) Prove or Disprove: If $n$ is an integer, then $\lfloor n \rfloor + \lceil n \rceil = 2n$.

   Notice that, by definition, for any integer $n$, then $\lfloor n \rfloor = n$. Similarly, for any integer $n$, $\lceil n \rceil = n$.

   Hence $\lfloor n \rfloor + \lceil n \rceil = n + n = 2n$, so the statement above is true.

   (b) (15 points) Let $a$ be an integer, Prove that if $3|n^2$, then $3|n$.

   **Proof:** Using contraposition, suppose that 3 does not divide $n$. Then $n = 3k + 1$ for some $k \in \mathbb{Z}$, or $n = 3k + 2$ for some $k \in \mathbb{Z}$ (its remainder upon division by zero must be either 1 or 2).

   **Case 1:** If $n = 3k + 1$ for some $k \in \mathbb{Z}$, then $n^2 = (3k+1)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$, so $n^2$ is not divisible by 3 (it would have remainder of 1).

   **Case 2:** If $n = 3k + 2$ for some $k \in \mathbb{Z}$, then $n^2 = (3k + 1)^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1$, so $n^2$ is not divisible by 3 (it would again have remainder of 1).

   Since both cases hold, this completes the proof. $\square$.