Math 311
Exam 1 - Practice Problem Solutions

1. For each of the following, translate the given argument into symbolic form. Then determine whether or not the given argument is valid by either writing a 2-column proof for the argument or by finding a counterexample.

   (a) If I work hard every day then I will get a promotion at work.
   If I do not get a promotion at work then I will not be able to afford my house payment.
   I can afford my house payment.
   ___
   Therefore I work hard every day.

   We assign variables: w : I work hard every day. p : I get a promotion at work. a : I can afford my house payment.

   The the symbolic form of this argument is:

   $w \to p$
   $\neg p \to \neg a$
   a
   ___
   $\therefore w$

   This argument is not valid. Notice that we can use the final two premises along with the Law of Contraposition to conclude p: this person got a promotion. However, concluding that this person worked hard in order to get a promotion, based on the first premise would be the Fallacy of the Converse. We know that working hard every day leads to getting a promotion, but we cannot assume that since they got a promotion, they must have worked hard every day. The problem truth values are when w is False, p is True, and a is True.

   (b) If gas is expensive and parking is inconvenient then I will take the bus to school.
   If I take the bus to school then I will not be able to take a night class.
   I am taking a night class.
   ___
   Therefore gas is not too expensive or parking is convenient.

   We begin by assigning variables as follows: g : gas is expensive. p : parking is inconvenient. b : I take the bus to school. n : I take a night class.

   Then the symbolic form of this argument is:

   $(g \wedge p) \to b$
   $b \to \neg n$
   n
   ___
   $\therefore \neg g \vee \neg p)$

   We now construct a 2-column proof as follows:

Statement	**Reason**
1. $b \to \neg n$	Premise
2. n	Premise
3. $\neg b$	Law of Contraposition (Modus Tollens) (1, 2)
4. $(g \wedge p) \to b$	Premise
5. $\neg(g \wedge p)$	Law of Contraposition (Modus Tollens) (3,4)
6. $\neg g \vee \neg p$	De Morgan's Law (5)

   Hence this argument is valid.

2. Determine whether or not the given argument is valid by translating it into symbolic form and identifying its form as a known valid argument form or a known fallacy.

   (a) If I want to go out on Saturday night then I need to study for my exam during the afternoon.
   I do not want to go out on Saturday night.
   ___
   Therefore I do not need to study for my exam during the afternoon.

   We assign variables: O : I want to go out on Saturday night. S : I need to study for my exam during the afternoon.

   The the symbolic form of this argument is:

   $O \to S$
   $\neg O$
   ___
   $\therefore \neg S$

   The form of this argument is the Fallacy of Denying the Hypothesis, so it is not valid.

(b) If I study for my exam Saturday during the afternoon then I will go out on Saturday night.
I do not go out on Saturday night.

Therefore I did not study for my exam Saturday afternoon.

We assign variables: O : I will go out on Saturday night. S : I study for my exam Saturday during the afternoon.

The the symbolic form of this argument is:

$S \to O$
$\neg O$

$\therefore \neg S$

The form of this argument is Modus Tollens, so it is valid.

(c) If I do not study for my exam then I will not get a good grade on it.
I got a good grade on my exam.

Therefore I studied for my exam.

We assign variables: S : I study for my exam. G : I get a good grade on my exam.

The the symbolic form of this argument is:

$\neg S \to \neg G$
G

$\therefore S$

The form of this argument is Modus Tollens, so it is valid.

3. Prove that each of the following arguments are valid by constructing a 2-column proof.

(a) If you are unhappy with the results of the election then the candidate you voted for did not win.
If the candidate you voted for did not win, then she will run again next election.
The candidate you voted for is not running next election

Therefore you are happy with the election results

We begin by translating this argument into symbolic form. Defining the variables H : You are happy with the results of the election. W : The candidate you voted for won. R : The candidate you voted for is running in the next election, we have the following argument:

(b) $\neg H \to \neg W$
$\neg W \to R$
$\neg R$

$\therefore H$

Then we can construct the following proof:

Statement	**Reason**
$\neg H \to \neg W$	Premise
$\neg W \to R$	Premise
$\neg H \to R$	Hypothetical Syllogism
$\neg R$	Premise
H	Modus Tollens

Hence this argument is valid.

(c) Everyone who is brave and intelligent majors in mathematics.
Tony is not a math major.
Tony is brave.

Therefore Tony is not intelligent.

We begin by translating this argument into symbolic form. Let x represent people. We then defining the predicates $B(x)$: Person x is brave. $I(x)$: Person x is intelligent. $M(x)$: Person x is a math major. Then we have the following argument:

(d) $\forall x\,[(B(x) \wedge I(x)) \to M(x)]$
$\neg M(\text{Tony})$
$B(\text{Tony})$

$\therefore \neg I(\text{Tony})$

Then we can construct the following proof:

Statement	**Reason**
$\forall x\,[(B(x) \wedge I(x)) \to M(x)]$	Premise
$(B(\text{Tony}) \wedge I(\text{Tony})) \to M(\text{Tony})$	Universal Instantiation
$\neg M(\text{Tony})$	Premise
$\neg\,(B(\text{Tony}) \wedge I(\text{Tony}))$	Modus Tollens
$\neg B(\text{Tony}) \vee \neg I(\text{Tony})$	DeMorgan's Law
$B(\text{Tony})$	Premise
$\neg I(\text{Tony})$	Disjunctive Syllogism

Hence this argument is valid.

4. Prove that the product of three odd numbers is odd.

   Proof:

   We will begin by proving the following Lemma:

   Lemma: The product of two odd numbers is odd.

   Proof: Let m and n be odd numbers. Then, by definition, there is an integer k such that $m = 2k + 1$ and there is an integer l such that $n = 2l + 1$.

   Then $mn = (2k + 1)(2l + 1) = 4kl + 2k + 2l + 1 = 2(2kl + k + l) + 1$, which, since $2kl + k + l$ is an integer, shows that mn is odd. □

   From this, we can prove the original theorem as follows:

   Let m, n and p be odd integers. Using the lemma, mn is odd. But then $mnp = (mn)p$ is the product of two odd integers, so again using the lemma, mnp is odd.

5. Prove that if n is an integer and that $n^2 + 11$ is even, then n is odd.

   Proof:

   We will proceed using Proof by Contraposition.

   Suppose that n is even. Then, by definition, $n = 2k$ for some integer k. Therefore, $n^2+11 = (2k)^2+11 = 4k^2+10+1 = 2(2k^2 + 5) + 1$.

   Thus $n^2 + 1$ is odd.

   We have shown that if n is odd, then $n^2 + 11$ is odd, hence the contrapositive of this statement is true.

   Therefore, if n is an integer and $n^2 + 11$ is even, then n is odd. □

6. Prove that $m^2 = n^2$ if and only if $m = n$ or $m = -n$.

   "$\Rightarrow$"

   We begin by proving the backward direction of this biconditional statement. Suppose that either $m = n$ or $m = -n$.

   If $m = n$, then $m^2 = n^2$ follows using basic principles of Algebra.

   Similarly, if $m = -n$, then $m^2 = (-n)^2 = n^2$ follows using basic principles of Algebra.

   "$\Leftarrow$"

   To prove the backward direction of this biconditional statement, suppose that $m^2 = n^2$.

   Notice that the square of any real number is non-negative, so we can take the square root of each side of this equation in order to obtain:

   $\sqrt{m^2} = \sqrt{n^2}$, or $|m| = |n|$.

   Thus, using basic principles of algebra, either $m = n$ or $m = -n$. □

7. Show that the following three statements are all equivalent:

   (i) x is rational

   (ii) $\frac{x}{2}$ is rational

   (iii) $x + 1$ is rational.

   Proof: To show that all of these statements are equivalent, we will show $(i) \rightarrow (ii) \rightarrow (iii) \rightarrow (i)$.

   "$(i) \rightarrow (ii)$"

   Assume that x is rational. Then, by definition, $x = \frac{a}{b}$ for integers a, b with $b \neq 0$.

   But then $\frac{x}{2} = \frac{a}{2b}$, so $\frac{x}{2}$ is also rational.

   "$(ii) \rightarrow (iii)$"

   Assume that $\frac{x}{2}$ is rational. Then, by definition, $\frac{x}{2} = \frac{a}{b}$ for integers a, b with $b \neq 0$.

   But then $2 \cdot \frac{x}{2} + 1 = x + 1 = \frac{2a}{b} + 1 = \frac{2a+b}{b}$, so $x + 1$ is also rational.

   "$(iii) \rightarrow (i)$"

   Assume that $x + 1$ is rational. Then, by definition, $x + 1 = \frac{a}{b}$ for integers a, b with $b \neq 0$.

   But then $x = (x + 1) - 1 = \frac{a}{b} - 1 = \frac{a-b}{b}$, so x is also rational. □

8. Prove or Disprove: The sum of two irrational numbers is irrational.

   This theorem is false. Here is a counterexample: as we proved in class, $\sqrt{2}$ is irrational. A similar proof shows that $-\sqrt{2}$ is irrational (otherwise, if $-\sqrt{2} = \frac{a}{b}$, then $\sqrt{2} = \frac{-a}{b}$. However, $\sqrt{2} + (-\sqrt{2}) = 0$ which is rational.

9. Prove that there is an integer m such that $m^3 > 10^{10}$. Is your proof constructive or non-constructive?

   Let $m = 2200$. Then $m^3 = (2200)^3 = 10,648,000,000 > 10^{10} = 10,000,000,000$.
   This proof is constructive.

10. Prove that given any two rational numbers $p < q$, there is a rational number r with $p < r < q$.

    Let p, q satisfy $p < q$. Let $r = \frac{p+q}{2}$.

    Claim 1: $p < r$.

    Since $p < q$, $q - p > 0$. Then $r = \frac{p+q}{2} = \frac{p+p+(q-p)}{2} = \frac{2p}{2} + \frac{q-p}{2} = p + \frac{q-p}{2} > p$.

    Claim 2: $r < q$.

    Since $q - p > 0$, $p - q < 0$. Then $r = \frac{p+q}{2} = \frac{q+q+(p-q)}{2} = \frac{2q}{2} + \frac{p-q}{2} = q + \frac{p-q}{2} < q$.

    Claim 3: r is rational.

    Since p and q are rational, $p = \frac{a}{b}$ and $q = \frac{c}{d}$ with a, b, c, d integers and $b, d \neq 0$.
    Then $r = \frac{\frac{a}{b}+\frac{c}{d}}{2} = \frac{\frac{ad+bc}{bd}}{2} = \frac{ad+bc}{2bd}$. Since $ad + bc$ is an integer and $2bd$ is a non-zero integer, then r is rational.

    Hence $r = \frac{p+q}{2}$ is a rational number satisfying $p < r < q$.

11. Prove that given a non-negative integer n, there is a unique non-negative integer m such that $m^2 \leq n < (m+1)^2$

    Let m be $\lfloor \sqrt{n} \rfloor$. Then $m \leq \sqrt{n}$, so $m^2 \leq m \cdot \sqrt{n} \leq \sqrt{n} \cdot \sqrt{n} = n$.
    Also, $m + 1 > \sqrt{n}$, so $(m+1)^2 > (m+1) \cdot \sqrt{n} > \sqrt{n} \cdot \sqrt{n} = n$.
    Hence $m^2 \leq n < (m+1)^2$.

    To see uniqueness, suppose k is an integer so that $k^2 \leq n < (k+1)^2$. Then $k \leq \sqrt{n}$ (since otherwise, if $k > \sqrt{n}$, then $k^2 > k\sqrt{n} > \sqrt{n} \cdot \sqrt{n} = n^2$).
    Similarly, $k + 1 > \sqrt{n}$. So we have $k \leq \sqrt{n} < k + 1$. Therefore, since both m and k are integers, we must have $m = k$.

12. Prove or disprove: Every non-negative integer can be written as the sum of at most 3 perfect squares.

    This statement is false. Notice that 7 cannot be written as the sum of 3 squares:
    $0+0+0=0$, $1+0+0=1$, $1+1+0=2$, $1+1+1=3$, $4+0+0=4$, $4+1+0=5$, $4+1+1=6$, $4+4+0=8$.

13. Which digits can occur as the final digit of the cube of an integer? Justify your answer.

    Notice that $0^3 = 0$, $1^3 = 1$, $2^3 = 8$, $3^3 = 27$, $4^3 = 64$, $5^3 = 125$, $6^3 = 216$, $7^3 = 343$, $8^3 = 512$, and $9^3 = 729$.

    From these examples, we see that any digit can occur as the final digit in the cube if an integer.

14. Use a paragraph to show that $(B - A) \cup (C - A) = (B \cup C) - A$.

Proof:

"$\subseteq$" Let $x \in (B - A) \cup (C - A)$. Then $x \in B - A$ or $x \in C - A$. We will consider these two cases separately.

Case 1: Suppose $x \in B - A$. Then, by definition of set difference, $x \in B$ and $x \notin A$. Since $x \in B$, then, by definition of set union, $x \in B \cup C$. Therefore, since $x \in B \cup C$ and $x \notin A$, we have $x \in (B \cup C) - A$.

Case 2: Suppose $x \in C - A$. As above, by definition of set difference, $x \in C$ and $x \notin A$. Since $x \in C$, then, by definition of set union, $x \in B \cup C$. Therefore, since $x \in B \cup C$ and $x \notin A$, we have $x \in (B \cup C) - A$.

Since these are the only possible cases, then $(B - A) \cup (C - A) \subseteq (B \cup C) - A$

"$\supseteq$" Let $x \in (B \cup C) - A$. Then, by definition of set difference, $x \in B \cup C$ and $x \notin A$. We will once again split into cases. Note that in both cases, we have $x \notin A$.

Case 1: Suppose $x \in B$. Then, since $x \notin A$, we have $x \in B - A$. Hence, by definition of set union, $x \in (B - A) \cup (C - A)$.

Case 2: Suppose $x \in C$. Then, since $x \notin A$, we have $x \in C - A$. Hence, by definition of set union, $x \in (B - A) \cup (C - A)$.

Since these are the only possible cases, $(B - A) \cup (C - A) \supseteq (B \cup C) - A$

Thus $(B - A) \cup (C - A) \subseteq (B \cup C) - A$. $\square$

15. Use a paragraph to show that $A \cup (A \cap B) = A$.

Proof:

"$\subseteq$" Let $x \in A \cup (A \cap B)$. Then, by definition of set union, $x \in A$ or $x \in A \cap B$. We will consider these two cases separately.

Case 1: Suppose $x \in A$. Then, the result is clear.

Case 2: Suppose $x \in A \cap B$. Then, by definition of set intersection, $x \in A$ and $x \in B$, so we again have the desired containment.

Hence $A \cup (A \cap B) \subseteq A$.

"$\supseteq$" Let $x \in A$. Then, by definition of set union, we must have $x \in A \cup (A \cap B)$. Hence $A \cup (A \cap B) \supseteq A$.

Thus $A \cup (A \cap B) = A$. $\square$

16. Use a paragraph proof to show that $A - B = A \cap \overline{B}$.

First, suppose $x \in A - B$. Then, by the definition of set subtraction, $x \in A$ and $x \notin B$. Thus $x \in \overline{B}$. Hence $x \in A \cap \overline{B}$. Therefore, $A - B \subseteq A \cap \overline{B}$.

Next, suppose $x \in A \cap \overline{B}$. Then $x \in A$ and $x \in \overline{B}$. Thus $x \notin B$. Hence $x \in A - B$. Therefore, $A \cap \overline{B} \subseteq A - B$.

Hence $A - B = A \cap \overline{B}$ $\square$.