Math 311
Exam 2 - Fall 2015 **Name:**______________________________

Instructions: You will have 55 minutes to complete this exam. The credit given on each problem will be proportional to the amount of correct work shown. Answers without supporting work will receive little credit.

1. Let $R = \{(1,1),(2,1),(2,2),(3,1),(1,3),(3,3)\}$ be a binary relation on the set $A = \{1,2,3\}$.

   (a) (4 points) Is R is reflexive? Justify.

   Yes. Notice that $(1,1)$, $(2,2)$, and $(3,3)$ are all elements of R.

   (b) (4 points) Is R symmetric? Justify.

   No. Notice that $(2,1) \in R$, but $(1,2) \notin R$.

   (c) (3 points) Find the matrix M_R representing this relation.

   $$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

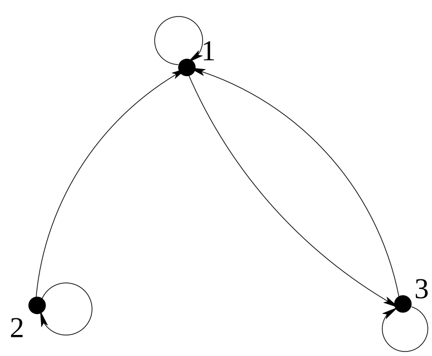
   (d) (5 points) Find the matrix representing $R \circ R$.

   $$M_{R\circ R} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

   (e) (4 points) Is R transitive? Justify your answer.

   No. From the matrix above, we see that the element $(2,3)$ is the culprit. In particular, $(2,1)$ and $(1,3)$ are in R but $(2,3)$ is not in R.

   (f) (3 points) Draw the Graph Γ_R representing the relation R.



2. (6 points) Let R be the relation on $A = \{0, 1, 2, 3, 4, 5\}$ given by $\{(1, 2), (3, 4), (5, 4)\}$. Find the smallest equivalence relation containing R. Give your answer as a set of ordered pairs in roster notation.

   Let S be the smallest equivalence relation containing R. Notice that the ordered pairs given suggest the following partition of the set A: $\{0\}$, $\{1, 2\}$, $\{3, 4, 5\}$

   Then $S = \{(0, 0), (1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (3, 5), (4, 3), (4, 4), (4, 5), (5, 3), (5, 4), (5, 5)\}$

3. (12 points) Let R be the relation on $\mathbb{Z}$ defined by $R = \{(m, n) \in \mathbb{Z} \times \mathbb{Z} : m + n \,\text{is even}\}$. Determine whether or not R is an equivalence relation. If R is an equivalence relation, describe its equivalence classes.

   Reflexive: Consider (n, n). Since $n + n = 2n$, which is even, then $(n, n) \in R$ for any $n \in \mathbb{Z}$. Hence R is reflexive.

   Symmetric: Suppose that $(m, n) \in R$. Then $m + n$ is even. However, $m + n = n + m$. Thus $n + m$ is also even. Therefore, $(n, m) \in R$. Hence R is symmetric.

   Transitive: Suppose that (m, n) and (n, p) are both in R. Then $m + n = 2k$ for some integer k and $n + p = 2\ell$ for some integer ℓ. Therefore, $(m+n)+(n+p) = 2k+2\ell$. That is, $m+2n+p = 2k+2\ell$. Thus $m+p = 2k+2\ell-2n = 2(k+\ell-n)$, so $m + p$ is even. Hence $(m, p) \in R$ and thus R is transitive.

   Since R is reflexive, symmetric, and transitive, then R is an equivalence relation. Recall that back in Chapter 1, we showed that the sum of two even integers is even, the sum of two odd integers is even, but the sum of two integers with opposite parity yields an odd result.

   From this, we see that R has two equivalence classes:

   $[0]_R = \{n \text{ is an even integer }\}$

   and

   $[1]_R = \{n \text{ is an odd integer }\}$

4. (10 points) Suppose that R and S are equivalence relations on a non-empty set A. Prove or disprove the statement: $R \cap S$ is an equivalence relation.

   Reflexivity: Let $a \in A$. Since R is an equivalence relation, we must have $(a, a) \in R$. Similarly, since S is an equivalence relation, we must have $(a, a) \in S$. Therefore, by definition of set intersection, $(a, a) \in R \cap S$. Hence $R \cap S$ is reflexive.

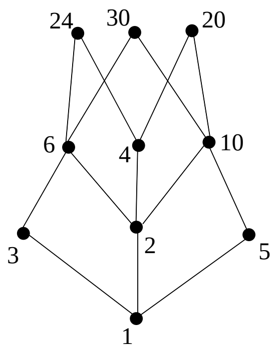
   Symmetricity: Suppose $(a, b) \in R \cap S$. Then, by definition of set intersection, we must have $(a, b) \in R$ and $(a, b) \in S$. Since R is an equivalence relation, R is symmetric, so we must have $(b, a) \in R$. Similarly, we must have $(b, a) \in S$. Hence $(b, a) \in R \cap S$, and hence $R \cap S$ is symmetric.

   Transitivity: Suppose $(a, b) \in R \cap S$ and $(b, c) \in R \cap S$. Then, by definition of set intersection, we must have $(a, b), (b, c) \in R$ and $(a, b), (b, c) \in S$. Since R is an equivalence relation, R is transitive, so we must have $(a, c) \in R$. Similarly, we must have $(a, c) \in S$. Hence $(a, c) \in R \cap S$, and hence $R \cap S$ is transitive.

   We have demonstrated that $R \cap S$ is reflexive, symmetric and transitive. Thus $R \cap S$ is an equivalence relation.

5. Given the poset $(\{1, 2, 3, 4, 5, 6, 10, 20, 24, 30\}, |)$

   (a) (8 points) Draw the Hasse Diagram for this poset.



   (b) (3 points) Find all maximal elements.

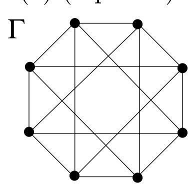
   From the diagram, we see that the maximal elements are: 20, 24, and 30.

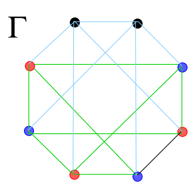
   (c) (3 points) Find the greatest lower bound of 6 and 10.

   The lower bounds of $\{6, 10\}$ are: 1 and 2. Therefore, the greatest lower bound is 2.

6. Determine whether of not each of the following graphs is planar. If a graph is planar, exhibit a planar drawing of the graph and verify that Euler's formula holds for this representation of the graph. If a graph is not planar, provide an argument that proves that the graph cannot be planar. [Note: Both of these graphs have only 8 vertices]

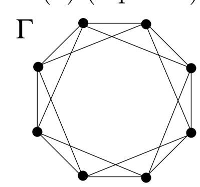
(a) (7 points)

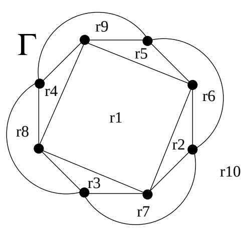




As shown in the drawing above,
this graph is has a $K_{3,3}$ subgraph.
So it is not planar by Kuratowski's Theorem.
Alternatively, notice that Γ has no
3-cycle, so applying Corollary 3 to Euler's Theorem
We should have $e \leq 2v - 4$ or
Since $16 > 2(8) - 4 = 12$, Γ is not planar

(b) (7 points)





The drawing above shows that this graph is
planar and has 10 regions.
Since $|E| = 16$ and $|V| = 8$
Verifying Euler's Theorem, $10 = 16 - 8 + 2$

7. Prove or Disprove:

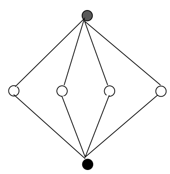
(a) (6 points) There is a planar simple connected graph with 6 vertices and 13 edges.

This statement is False. One way is to apply Corollary 1 to Euler's Formula, which states that if e is the number of edges and v the number of vertices in a connected planar graph, then $e \leq 3v - 6$. Here, $3v - 6 = 3(6) - 6 = 18 - 6 = 12$. Hence a simple connected planar graph with 6 vertices can have at most 12 edges.

(b) (6 points) There is a planar simple bipartite connected graph with 6 vertices and 8 edges.

This statement is True. The following example verifies this claim.

Γ



8. Given the Boolean function: $F(x, y, z) = \overline{x}\,(y + \overline{z}) + \overline{y}\,(\overline{x} + z)$

(a) (3 points) Find the value of $F(0, 1, 0)$.

$F(0,1,0) = \overline{0}\left(1+\overline{0}\right) + \overline{1}\left(\overline{0}+0\right) = 1 \cdot (1+1) + 0(1+0) = 1 \cdot 1 + 0(1) = 1 + 0 = 1$

(b) (8 points) Find the sum of products expansion for $F(x, y, z)$.

x	y	z	$\overline{x}$	$\overline{y}$	$\overline{z}$	$y+\overline{z}$	$\overline{x}+z$	$\overline{x}(y+\overline{z})$	$\overline{y}(\overline{x}+z)$	$\overline{x}\,(y+\overline{z})+\overline{y}\,(\overline{x}+z)$
0	0	0	1	1	1	1	1	1	1	1
0	0	1	1	1	0	0	1	0	1	1
0	1	0	1	0	1	1	1	1	0	1
0	1	1	1	0	0	1	1	1	0	1
1	0	0	0	1	1	1	0	0	0	0
1	0	1	0	1	0	0	1	0	1	1
1	1	0	0	0	1	1	0	0	0	0
1	1	1	0	0	0	1	1	0	0	0

Using the value table for this Boolean Function, we construct the sum of products expansion by including the minterms associated with each 1 present in the final column of the table:

$F(x, y, z) = \overline{x}\,\overline{y}\,\overline{z} + \overline{x}\,\overline{y}z + \overline{x}y\overline{z} + \overline{x}yz + x\overline{y}z.$

Note: We can also find this expression *without* constructing the table if we apply Boolean operations and equivalences to the original expression.

$F(x, y, z) = \overline{x}\,(y + \overline{z}) + \overline{y}\,(\overline{x} + z) = \overline{x}y + \overline{x}\,\overline{z} + \overline{x}\,\overline{y} + \overline{y}z = \overline{x}y(z + \overline{z}) + \overline{x}\,\overline{z}(y + \overline{y}) + \overline{x}\,\overline{y}(z + \overline{z}) + \overline{y}z(x + \overline{x}) = \overline{x}yz + \overline{x}y\overline{z} + \overline{x}\,\overline{z}y + \overline{x}\,\overline{z}\,\overline{y} + \overline{x}\,\overline{y}z + \overline{x}\,\overline{y}\,\overline{z} + \overline{y}zx + \overline{y}z\overline{x} = \overline{x}\,\overline{y}\,\overline{z} + \overline{x}\,\overline{y}z + \overline{x}y\overline{z} + \overline{x}yz + x\overline{y}z$

Extra Credit: (10 points) **Recall The Abstract Definition of a Boolean Algebra:** A **Boolean Algebra** is a set B with two binary operations $\vee$ and $\wedge$, elements 0 and 1, and a unary operation $\overline{}$ such that the following properties hold for all x, y and $z \in B$:

Property	**Name**
$x \vee 0 = x$ $x \wedge 1 = x$	Identity Laws
$x \vee \overline{x} = 1$ $x \wedge \overline{x} = 0$	Complement Laws
$(x \vee y) \vee z = x \vee (y \vee z)$ $(x \wedge y) \wedge z = x \wedge (y \wedge z)$	Associative Laws
$x \vee y = y \vee x$ $x \wedge y = y \wedge x$	Commutative Laws
$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$	Distributive Laws

Use a 2-column proof involving *only* the formal definition and properties given above to prove the Idempotent Law: $x = x \vee x$

Statement	**Reason**
(1) x	Given
(2) $x \vee 0$	Identity Law
(3) $x \vee (x \wedge \overline{x})$	Complement Law
(4) $(x \vee x) \wedge (x \vee \overline{x})$	Distributive Law
(5) $(x \vee x) \wedge 1$	Complement Law
(6) $x \vee x$	Identity Law