Math 311
Exam 2 - Practice Problems

1. Given the relation $R = \{(1,1),(1,3),(1,4),(2,2),(3,1),(3,4),(4,1),(4,3)\}$ on the set $A = \{1,2,3,4\}$:

   (a) Determine whether or not R is reflexive.
   (b) Determine whether or not R is irreflexive.
   (c) Determine whether or not R is symmetric.
   (d) Determine whether or not R is antisymmetric.
   (e) Determine whether or not R is transitive.

2. Given the relation $S = \{(1,1),(1,3),(2,1),(2,2),(2,3),(3,3),(4,4)\}$ on the set $A = \{1,2,3,4\}$:

   (a) Determine whether or not S is reflexive.
   (b) Determine whether or not R is irreflexive.
   (c) Determine whether or not R is symmetric.
   (d) Determine whether or not R is antisymmetric.
   (e) Determine whether or not R is transitive.

3. Suppose that R and S are symmetric relations on a non-empty set A. Prove or disprove each of these statements:

   (a) $R \cup S$ is symmetric.
   (b) $R \cap S$ is symmetric.
   (c) $R - S$ is symmetric.
   (d) $R \oplus S$ is symmetric.
   (e) $S \circ R$ is symmetric.

4. Given the relation $R = \{(1,1),(1,3),(1,4),(2,2),(3,1),(3,4),(4,1),(4,3)\}$ on the set $A = \{1,2,3,4\}$:

   (a) Find the matrix representation M_R for this relation. (b) Draw the graph representation of this relation Γ_R.

5. Given the relation $S = \{(1,1),(1,3),(2,1),(2,2),(2,3),(3,3),(4,4)\}$ on the set $A = \{1,2,3,4\}$:

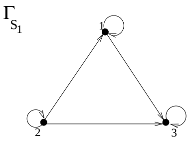
   (a) Find the matrix representation M_S for this relation. (b) Draw the graph representation of this relation Γ_S.

6. Let R_1 and R_2 be given by the matrices $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

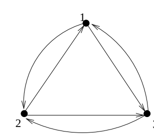
   (a) Determine whether or not R_1 is reflexive, irreflexive, symmetric, antisymmetric, transitive.
   (b) Determine whether or not R_2 is reflexive, irreflexive, symmetric, antisymmetric, transitive.
   (c) Find the matrices representing $\overline{R_1}$, $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 \oplus R_2$, and $R_1 \circ R_2$
   (d) Draw the graphs representing R_1 and R_2.

7. Given the graphs representing the relations S_1 and S_2:

   Γ_{S_1}

   Γ_{S_2}

   (a) Determine whether or not R_1 is reflexive, irreflexive, symmetric, antisymmetric, transitive.
   (b) Determine whether or not R_2 is reflexive, irreflexive, symmetric, antisymmetric, transitive.
   (c) Draw the graph representing $\overline{S_2}$, $S_1 \cup S_2$, $S_1 \cap S_2$, $S_2 - S_1$, and $S_2 \circ S_1$
   (d) Find the matrix representing S_1.
   (e) List the ordered pairs in S_2.

8. Determine whether or not the following binary relations are equivalence relations. Be sure to justify your answers.

   (a) $\{(0,0),(0,3),(0,4),(1,1),(1,2),(2,1),(2,2),(3,0),(3,3),(3,4),(4,0),(4,3),(4.4)\}$ on the set $A = \{0,1,2,3,4\}$
   (b) $\{(a,a),(a,b),(b,a),(b,b),(b,d),(c,c),(d,b),(d,d)\}$ $A = \{a,b,c,d\}$
   (c) $\{(x,y) \mid y$ is a biological parent of $x\}$ on the set of all people.
   (d) $\{(x,y) \in \mathbb{N} \times \mathbb{N} \mid lcm(x,y) = 10\}$
   (e) $\{(x,y) \in \mathbb{R} \times \mathbb{R} \mid y = x^2 + 1\}$
   (f) for each of (a)-(e) that **are** equivalence relations, find the equivalence classes for the relation.

9. Define a relation R on $\mathbb{R}^2$ by $\{((x_1, y_1), (x_2, y_2)) \,|\, (x_1^2 + y_1^2) = (x_2^2 + y_2^2)\}$

   (a) Show that R is an equivalence relation.

   (b) Describe the equivalence classes of R.

10. Given that $A = \{0, 1, 2, 3, 4\}$

   (a) Find the smallest equivalence relation on A containing the ordered pairs $\{(1, 1), (1, 2), (3, 4), (4, 0)\}$
   (b) Draw the graph of the equivalence relation for found in part (a).
   (c) List the equivalence classes of the relation for found in (a).

11. For each of the following collections of subsets of $A = \{1, 2, 3, 4, 5\}$, determine whether of not the collection is a partition. If it is, list the ordered pairs in the equivalence relation determined by the partition.

   (a) $\{\{1, 2\}, \{3, 4\}, \{5\}\}$
   (b) $\{\{1, 2, 4\}, \{3\}, \{5\}\}$
   (c) $\{\{1, 2, 3, 4\}, \{5\}\}$
   (d) $\{\{1, 2\}, \{3\}, \{5\}\}$
   (e) $\{\{1, 2\}, \{2, 3, 4\}, \{5\}\}$

12. Determine whether or not the following binary relations are partial orders. Be sure to justify your answers.

   (a) $\{(0, 0), (0, 1), (0, 2), (0, 3), (0, 4), (1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (3, 4), (4, 4)\}$ on the set $A = \{0, 1, 2, 3, 4\}$
   (b) $\{(a, a), (a, d), (b, b), (b, d), (c, c), (c, d), (d, d)\}$ on the set $A = \{a, b, c, d\}$
   (c) $\{(x, y) \,|\, y$ is a biological parent of $x\}$
   (d) $\{(x, y) \in \mathbb{R} \times \mathbb{R} \,|\, y = x^2 + 1\}$
   (e) for each of (a)-(d) that are posets, draw the Hasse diagram for the poset (for those that are defined on infinite sets, only draw a finite subpart of the diagram).

13. Indicate which element is greater for each given pair using the standard lexicographic ordering.

   (a) $(2, 7)$ and $(3, 4)$
   (b) $(2, 7, 4, 9)$ and $(2, 4, 7, 9)$
   (c) (a, c, e, d) and (i, c, e, d)
   (d) (b, a, n, d, a, n, a) and (b, a, n, a, n, a, s)

14. Draw the Hasse Diagram for the poset $(\mathcal{P}(\{1, 2, 3\}), \supseteq)$

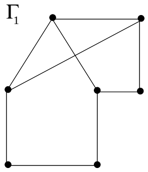
15. Draw the Hasse Diagram for the poset $(\mathcal{P}(\{0, 1, 2, 3\}), \subseteq)$

16. Given the poset $(\{1, 2, 3, 5, 6, 7, 10, 20, 30, 60, 70\}, |)$

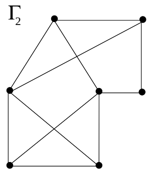
   (a) Draw the Hasse Diagram for this poset.
   (b) Find the maximal elements.
   (c) Find the minimal elements.
   (d) Find the greatest element or explain why there is no greatest element.
   (e) Find the least element or explain why there is no least element.
   (f) Find all upper bounds of $\{2, 5\}$
   (g) Find the least upper bound of $\{2, 5\}$ (if it exists).
   (h) Find all lower bounds of $\{6, 10\}$
   (i) Find the greatest lower bound of $\{6, 10\}$ (if it exists).

17. Determine whether of not each of the following graphs is planar. If a graph is planar, exhibit a planar drawing of the graph and verify that Euler's formula holds for this representation of the graph. If a graph is not planar, provide an argument that proves that the graph cannot be planar.

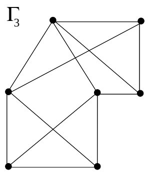
Γ_1



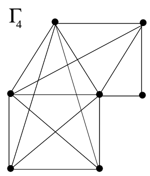
Γ_2



Γ_3



Γ_4



18. For each description given, either draw a planar graph that meets the description or prove that no planar graph can meet the description given.

    (a) A simple graph with 5 vertices and 8 edges.
    (b) A simple graph with 6 vertices and 13 edges.
    (c) A simple bipartite graph with 7 vertices and 10 edges
    (d) A simple bipartite graph with 7 vertices and 11 edges.

19. Find the value of the following Boolean expressions:

    (a) $1 \cdot \overline{(1+0)} + \overline{0}(1+\overline{0})$
    (b) $\overline{[\overline{1} + (\overline{0} \cdot 1)]} + [\overline{0} + \overline{0} \cdot 1]$

20. Build a value table for the following Boolean functions:

    (a) $F(x, y) = x + \overline{x}y$
    (b) $F(x, y, z) = xyz + y(\overline{x} + \overline{z})$

21. Use value tables to determine whether of not the following pairs of Boolean Expressions are equivalent:

    (a) $\overline{x} + \overline{y}$ and $\overline{xy}$
    (b) $\overline{xyz}$ and $\overline{x} + \overline{y} + \overline{z}$

22. Use a 2-column proof to prove each of the following:

    (a) $(xyz) + (yz) = yz$
    (b) $\overline{(x+z) \cdot (\overline{y}+z)} = (\overline{x} + y) \cdot \overline{z}$

23. Given the following value table:

x	y	z	$F(x, y, z)$	$G(x, y, z)$	$H(x, y, z)$
0	0	0	0	1	0
0	0	1	0	0	1
0	1	0	1	0	1
0	1	1	0	1	0
1	0	0	1	0	1
1	0	1	0	1	0
1	1	0	0	0	0
1	1	1	0	0	1

    (a) Find the sum of products expansion for $F(x, y, z)$
    (b) Find the sum of products expansion for $G(x, y, z)$
    (c) Find the sum of products expansion for $H(x, y, z)$

24. Find the sum of products expansion for each of the following Boolean Functions:

    (a) $F(x, y, z) = x + \overline{y} + x\overline{z}$
    (b) $F(x, y, z) = (x + \overline{y})z + x(\overline{y} + z)$
    (c) $F(x, y, z) = xy + \overline{x}y + \overline{y}$
    (d) $F(w, x, y, z) = (x + y)(z + \overline{w})$

25. Find the sum of products expansion of a Boolean function $F(s, t, x, y, z)$ that has value 1 if and only if an even number of the variables have value 1.

26. Given $F(x, y, z) = x(\overline{y} + z)$

    (a) Express $F(x, y, z)$ as a Boolean expression using only the operations $\cdot$ and $\overline{}$.
    (b) Express $F(x, y, z)$ as a Boolean expression using only the operations $+$ and $\overline{}$.
    (c) Express $F(x, y, z)$ as a Boolean expression using only the operation $|$.
    (d) Express $F(x, y, z)$ as a Boolean expression using only the operation $\downarrow$.

27. Using *only* the abstract definition of a Boolean Algebra, prove the following:

    (a) Prove that the law of the double complement holds. That it, that $\overline{\overline{x}} = x$ for every element x.
    (b) Prove that De Morgan's Laws hold. That is, that for all x, y, that $\overline{x \vee y} = \overline{x} \wedge \overline{y}$ and $\overline{x \wedge y} = \overline{x} \vee \overline{y}$