Math 311
Exam 2 - Practice Problems

1. Given the relation $R = \{(1,1),(1,3),(1,4),(2,2),(3,1),(3,4),(4,1),(4,3)\}$ on the set $A = \{1,2,3,4\}$:

   (a) Determine whether or not R is reflexive.

   R is not reflexive since $(4,4) \notin R$.

   (b) Determine whether or not R is irreflexive.

   R is not irreflexive since $(1,1) \in R$.

   (c) Determine whether or not R is symmetric.

   R is symmetric, since the reverse of every ordered pair is an element of R.

   (d) Determine whether or not R is antisymmetric.

   Since their are several pairs of non-reflexive symmetric elements in R, [for example, (1,3) and (3,1)] R is not antisymmetric.

   (e) Determine whether or not R is transitive.

   R is not transitive. Notice that $(4,1)$ and $(1,4)$ are in R, but $(4,4) \notin R$.

2. Given the relation $S = \{(1,1),(1,3),(2,1),(2,2),(2,3),(3,3),(4,4)\}$ on the set $A = \{1,2,3,4\}$:

   (a) Determine whether or not S is reflexive.

   S is reflexive since $(1,1)$, $(2,2)$, $(3,3)$ and $(4,4)$ are all elements of S.

   (b) Determine whether or not R is irreflexive.

   Since S is reflexive, it cannot be irreflexive, since $(a,a) \in S$ for all a.

   (c) Determine whether or not R is symmetric.

   S is not symmetric, since $(1,3) \in R$ and $(3,1) \notin S$.

   (d) Determine whether or not R is antisymmetric.

   We see that for every non-reflexive element in S, its reverse is not in S, so S is antisymmetric.

   (e) Determine whether or not R is transitive.

   S is transitive. The only non-reflexive pair in S that overlaps is $(2,1)$ and $(1,3)$, and we see that $(2,3) \in S$.

3. Suppose that R and S are symmetric relations on a non-empty set A. Prove or disprove each of these statements:

   (a) $R \cup S$ is symmetric.
   Let $(a,b) \in R \cup S$. Then $(a,b) \in R$ or $(a,b) \in S$.
   If $(a,b) \in R$, since R is symmetric, $(b,a) \in R$, so $(b,a) \in R \cup S$.
   Similarly, if $(a,b) \in S$, since S is symmetric, $(b,a) \in S$, so $(b,a) \in R \cup S$.
   Therefore $R \cup S$ is symmetric.

   (b) $R \cap S$ is symmetric.
   Let $(a,b) \in R \cap S$.
   Then $(a,b) \in R$, so since R is symmetric, $(b,a) \in R$. Similarly, $(a,b) \in S$, so since S is symmetric, $(b,a) \in S$.
   Thus $(b,a) \in R \cap S$. Therefore $R \cap S$ is symmetric.

   (c) $R - S$ is symmetric.
   Let $(a,b) \in R - S$.
   Then $(a,b) \in R$ and $(a,b) \notin S$. Since $(a,b) \in R$, and R is symmetric, then $(b,a) \in R$.
   Suppose $(b,a) \in S$. Since S is symmetric, then $(a,b) \in S$. But we assumed that $(a,b) \notin S$, so this is impossible. That is, if $(a,b) \notin S$, then $(b,a) \notin S$.
   Thus $(b,a) \in R - S$, and hence $R - S$ is symmetric.

   (d) $R \oplus S$ is symmetric.
   Recall that $R \oplus S = (R - S) \cup (S - R)$. From part (c) above, If R is symmetric and S is symmetric then $R - S$ is symmetric. By the same argument, $S - R$ is symmetric.
   Finally, by part (a) above, the union of two symmetric sets is symmetric. Thus $(R - S) \cup (S - R) = R \oplus S$ is symmetric.

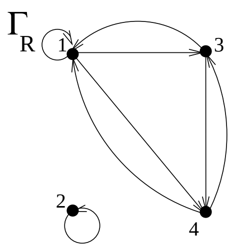
   (e) $S \circ R$ is symmetric.
   This statement is false. To see this, consider the example: $R = \{(1,2),(2,1)\}$ and $S = \{(2,3),(3,2)\}$. Notice that both R and S are symmetric. However, $S \circ R = \{(1,3)\}$ [Recall that a pair (a,c) is in $S \circ R$ if there is a b such that $(a,b) \in R$ and $(b,c) \in S$] Therefore, $S \circ R$ is not symmetric.

4. Given the relation $R = \{(1,1),(1,3),(1,4),(2,2),(3,1),(3,4),(4,1),(4,3)\}$ on the set $A = \{1,2,3,4\}$:

(a) Find the matrix representation M_R for this relation.

$$M_R = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

(b) Draw the graph representation of this relation Γ_R.

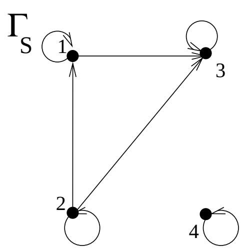


5. Given the relation $S = \{(1,1),(1,3),(2,1),(2,2),(2,3),(3,3),(4,4)\}$ on the set $A = \{1,2,3,4\}$:

(a) Find the matrix representation M_S for this relation.

$$M_S = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) Draw the graph representation of this relation Γ_S.



6. Let R_1 and R_2 be given by the matrices $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

(a) Determine whether or not R_1 is reflexive, irreflexive, symmetric, antisymmetric, transitive.

Since R_1 has $1s$ along the main diagonal, R_1 is reflexive and so R is not irreflexive.

R_1 is not symmetric since the matrix is not symmetric. In fact, R_1 is antisymmetric since every non-diagonal symmetric pair consists of a 1 and a 0

R_1 is transitive [the only non-reflexive overlapping pair is $(1,3)$ and $(3,2)$, and $(1,2) \in R_1$]

(b) Determine whether or not R_2 is reflexive, irreflexive, symmetric, antisymmetric, transitive.

Since R_2 has $0s$ along the main diagonal, R_1 is irreflexive and so R is not reflexive.

R_2 is symmetric since the matrix is symmetric, hence R_2 is not antisymmetric.

R_1 is not transitive [$(1,2) \in R_2$ and $(2,1) \in R_2$, but $(2,2) \notin R_2$]

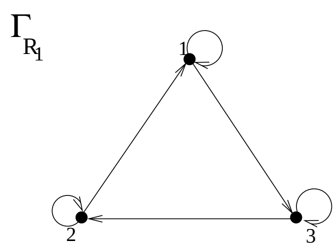
(c) Find the matrices representing $\overline{R_1}$, $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 \oplus R_2$, and $R_1 \circ R_2$

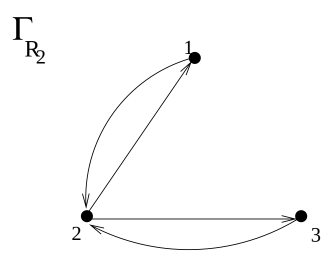
$$M_{\overline{R_1}} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad M_{\overline{R_1 \cup R_2}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

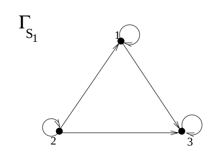
$$M_{\overline{R_1 \cap R_2}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad M_{\overline{R_1 \oplus R_2}} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

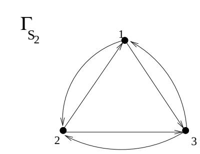
$$M_{\overline{R_1 \circ R_2}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \odot \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

(d) Draw the graphs representing R_1 and R_2.









7. Given the graphs representing the relations S_1 and S_2:

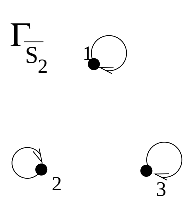
   (a) Determine whether or not R_1 is reflexive, irreflexive, symmetric, antisymmetric, transitive.

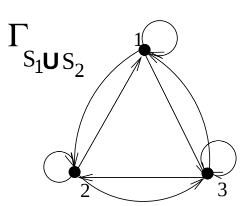
   From the graph, we see that S_1 is reflexive (there are loops at each vertex) , antisymmetric (none of the non-loop edges have opposite edges present) , and transitive (there is only one non-trivial edge to check).

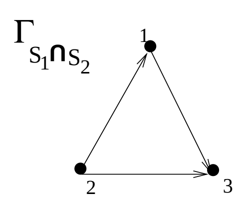
   (b) Determine whether or not R_2 is reflexive, irreflexive, symmetric, antisymmetric, transitive.

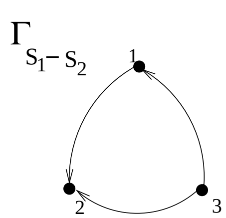
   From the graph, we see that S_2 is irreflexive (none of the vertices have loop edges), symmetric (each non-loop edge has on opposite), and not transitive (note that since we have symmetric pairs of edges, we would need loop edges to be present for this relation to be transitive. For example, $(1,2) \in R$ and $(2,1) \in R$ but $(1,1) \notin R$).

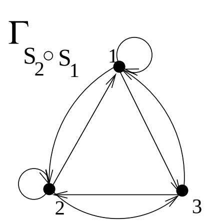
   (c) Draw the graph representing $\overline{S_2}$, $S_1 \cup S_2$, $S_1 \cap S_2$, $S_2 - S_1$, and $S_2 \circ S_1$











(d) Find the matrix representing S_1.

$$M_{S_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

(e) List the ordered pairs in S_2.
$S_2 = \{(1,2),(2,1),(2,3),(3,2),(1,3),(3,1)\}$

8. Determine whether or not the following binary relations are equivalence relations. Be sure to justify your answers.

(a) $\{(0,0),(0,3),(0,4),(1,1),(1,2),(2,1),(2,2),(3,0),(3,3),(3,4),(4,0),(4,3),(4.4)\}$ on the set $A = \{0,1,2,3,4\}$

First notice that this relation is reflexive, since $(0,0),(1,1),(2,2),(3,3),(4,4) \in R$.
Next, one can also see that this relation is symmetric, since the reverse of every non-reflexive element is present.
Finally, one can verify that this relation is also transitive by checking all overlapping pairs.
Thus this relation is an equivalence relation.

(b) $\{(a,a),(a,b),(b,a),(b,b),(b,d),(c,c),(d,b),(d,d)\}$ $A = \{a,b,c,d\}$

First notice that this relation is reflexive, since $(a,a),(b,b),(c,c),(d,d) \in R$.
Next, one can also see that this relation is symmetric, since the reverse of every non-reflexive element is present.
However, this relation is not transitive, since (a,b) and (b,d) are in R but $(a,d) \notin R$.
Thus this relation is not an equivalence relation.

(c) $\{(x,y) \mid y$ is a biological parent of $x\}$ on the set of all people.

This relation is not reflexive, since a person is not a parent of themselves.
It is also not symmetric, since if a is a parent of b, then b is not a parent of b
Finally, transitivity also fails, since grandparents are not parents.
Thus this relation is not an equivalence relation.

(d) $\{(x,y) \in \mathbb{N} \times \mathbb{N} \mid lcm(x,y) = 10\}$

Notice that this relation is given by $\{(1,10),(2,5),(2,10),(5,2),(5,10),(10,1),(10,2),(10,5),(10,10)\}$
This relation is symmetric, but it is not reflexive since $(2,2) \notin R$, and it is not transitive since $(5,10),(10,1) \in R$ but $(5,1) \notin R$
Thus this relation is not an equivalence relation.

(e) $\{(x,y) \in \mathbb{R} \times \mathbb{R} \mid y = x^2 + 1\}$

This relation is not reflexive since $(-1,-1) \notin R$.
This relation is also not symmetric since $(2,5) \in R$, but $(5,2) \notin R$.
This relation is not transitive since $(1,2),(2,5) \in R$, but $(1,5) \notin R$
Thus this relation is not an equivalence relation.

(f) for each of (a)-(e) that **are** equivalence relations, find the equivalence classes for the relation.

Notice that (a) is the only equivalence relation. It equivalence classes are $[0]_R = \{0,3,4\}$ and $[1]_R = \{1,2\}$.

9. Define a relation R on $\mathbb{R}^2$ by $\{((x_1,y_1),(x_2,y_2)) \mid (x_1^2 + y_1^2) = (x_2^2 + y_2^2)\}$

(a) Show that R is an equivalence relation.

First notice that this relation is reflexive, since for every ordered pair (x_1,y_1), $(x_1^2+y_1^2) = (x_1^2+y_1^2)$.
Next, one can also see that this relation is symmetric, since if $(x_1^2+y_1^2) = (x_2^2+y_2^2)$ then $(x_2^2+y_2^2) = (x_1^2+y_2^2)$
Finally, one can verify that this relation is also transitive by observing that if $(x_1^2+y_1^2) = (x_2^2+y_2^2)$ and $(x_2^2+y_2^2) = (x_3^2+y_3^2)$, then $(x_1^2+y_1^2) = (x_3^2+y_3^2)$.
Thus this relation is an equivalence relation.

(b) Describe the equivalence classes of R.

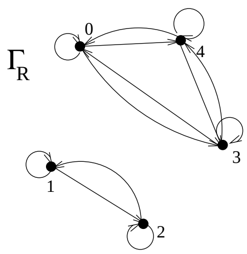
Suppose $(x_1^2 + y_1^2) = r^2$ for some $r \in \mathbb{R}$. then $[(x_1, y_1)]_R = \{(x_i, y_i) \mid (x_i^2 + y_i^2) = r^2\}$.
That is, equivalence classes are sets of points that lie on the same circle centered about the origin.

10. Given that $A = \{0, 1, 2, 3, 4\}$

(a) Find the smallest equivalence relation on A containing the ordered pairs $\{(1,1), (1,2), (3,4), (4,0)\}$

The smallest equivalence relation containing the given points is:
$\{(0,0), (0,3), (0,4), (1,1), (1,2), (2,1), (2,2), (3,0), (3,3), (3,4), (4,0), (4,3), (4,4)\}$

(b) Draw the graph of the equivalence relation for found in part (a).



(c) List the equivalence classes of the relation for found in (a).

The equivalence classes are: $[0]_R = \{0, 3, 4\}$ and $[1]_R = \{1, 2\}$.

11. For each of the following collections of subsets of $A = \{1, 2, 3, 4, 5\}$, determine whether of not the collection is a partition. If it is, list the ordered pairs in the equivalence relation determined by the partition.

(a) $\{\{1, 2\}, \{3, 4\}, \{5\}\}$

This collection of subsets is a partition. Notice that the sets are disjoint and that their union is all of A.

The ordered pairs in the equivalence relation generated by this partition are:
$\{(1,1), (1,2), (2,1), (2,2), (3,3), (3,4), (4,3), (4,5), (5,5)\}$

(b) $\{\{1, 2, 4\}, \{3\}, \{5\}\}$

This collection of subsets is a partition. Notice that the sets are disjoint and that their union is all of A.

The ordered pairs in the equivalence relation generated by this partition are:
$\{(1,1), (1,2), (1,4), (2,1), (2,2), (2,4), (4,1), (4,2), (4,4)(3,3), (5,5)\}$

(c) $\{\{1, 2, 3, 4\}, \{5\}\}$

This collection of subsets is a partition. Notice that the sets are disjoint and that their union is all of A.

The ordered pairs in the equivalence relation generated by this partition are:
$\{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4), (4,1), (4,2), (4,3), (4,4), (5,5)\}$

(d) $\{\{1, 2\}, \{3\}, \{5\}\}$

This collection of subsets is not a partition. The sets are disjoint but their union is missing 4.

(e) $\{\{1, 2\}, \{2, 3, 4\}, \{5\}\}$

This collection of subsets is a not partition. Notice that the sets are not disjoint since two of the sets contain 2.

12. Determine whether or not the following binary relations are partial orders. Be sure to justify your answers.

(a) $\{(0,0),(0,1),(0,2),(0,3),(0,4),(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(3,4),(4,4)\}$ on the set $A = \{0,1,2,3,4\}$

Notice that this relation is reflexive, since it contains all reflexive pairs.
This relation is also antisymmetric, since there are no reverses of non-reflexive elements present in the relation.
Finally, this relation is transitive [check all overlapping pairs].
Thus this is a partial order.

(b) $\{(a,a),(a,d),(b,b),(b,d),(c,c),(c,d),(d,d)\}$ on the set $A = \{a,b,c,d\}$

Notice that this relation is reflexive, since it contains all reflexive pairs.
This relation is also antisymmetric, since there are no reverses of non-reflexive elements present in the relation.
Finally, this relation is transitive [in fact, the only overlapping pairs each involve a reflexive pair].
Thus this is a partial order.

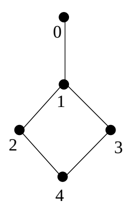
(c) $\{(x,y) \mid y \text{ is a biological parent of } x\}$

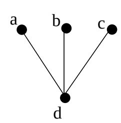
This relation is not reflexive since a person is not their own parent, so it is not a partial order.

(d) $\{(x,y) \in \mathbb{R} \times \mathbb{R} \mid y = x^2 + 1\}$

This relation is not reflexive since $(-1,-1) \notin R$, so it is not a partial order.

(e) for each of (a)-(d) that are posets, draw the Hasse diagram for the poset (for those that are defined on infinite sets, only draw a finite subpart of the diagram).

(a) 

(b) 

13. Indicate which element is greater for each given pair using the standard lexicographic ordering.

(a) $(2,7)$ and $(3,4)$

$(2,7) \prec (3,4)$ since $2 < 3$.

(b) $(2,7,4,9)$ and $(2,4,7,9)$

$(2,4,7,9) \prec (2,7,4,9)$ since $2 = 2$ and $4 < 7$.

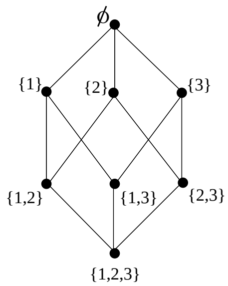
(c) (a,c,e,d) and (i,c,e,d)

$(a,c,e,d) \prec (i,c,e,d)$ since $a < i$.

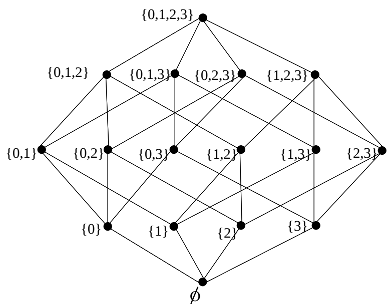
(d) (b,a,n,d,a,n,a) and (b,a,n,a,n,a,s)

$(b,a,n,a,n,a,s) \prec (b,a,n,d,a,n,a)$ since $b = b$, $a = a$, $n = n$, and $a < d$.

14. Draw the Hasse Diagram for the poset $(\mathcal{P}(\{1,2,3\}), \supseteq)$

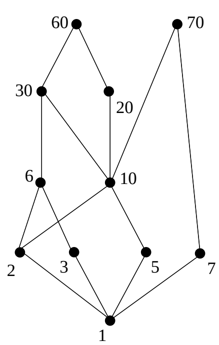


15. Draw the Hasse Diagram for the poset $(\mathcal{P}(\{0,1,2,3\}), \subseteq)$



16. Given the poset $(\{1,2,3,5,6,7,10,20,30,60,70\}, |)$

(a) Draw the Hasse Diagram for this poset.



(b) Find the maximal elements.

60, 70

(c) Find the minimal elements.

1

(d) Find the greatest element or explain why there is no greatest element.

There is no greatest element, since there are two maximal elements.

(e) Find the least element or explain why there is no least element.

1 is the least element

(f) Find all upper bounds of $\{2,5\}$

10, 20, 30, 60, 70

(g) Find the least upper bound of $\{2,5\}$ (if it exists).

10 is the least upper bound.

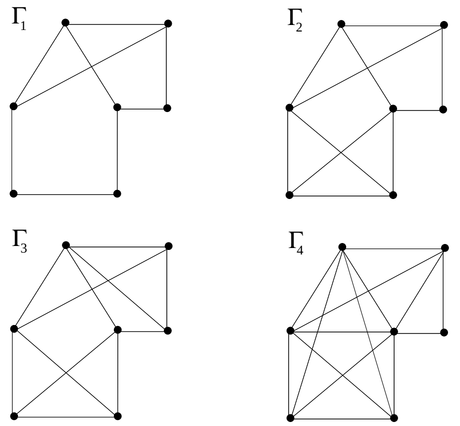
(h) Find all lower bounds of $\{6,10\}$

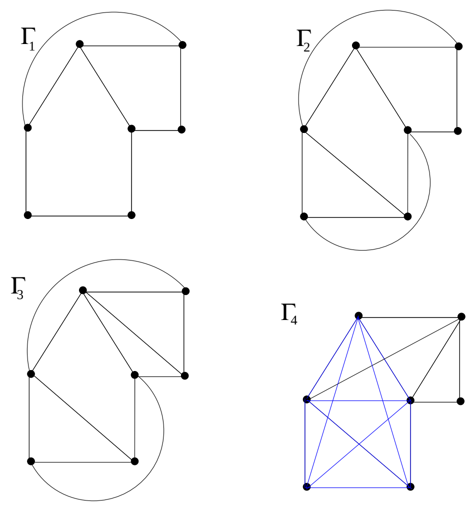
2, 1

(i) Find the greatest lower bound of $\{6,10\}$ (if it exists).

2 is the greatest lower bound.

17. Determine whether of not each of the following graphs is planar. If a graph is planar, exhibit a planar drawing of the graph and verify that Euler's formula holds for this representation of the graph. If a graph is not planar, provide an argument that proves that the graph cannot be planar.





Graphs 1-3 are planar, as shown by the diagram above.

For Γ_1, $v = 7$, $e = 9$, and $r = 4$, which satisfies $r = e - v + 2$, since $9 - 7 + 2 = 4$.

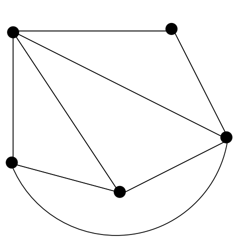
For Γ_2, $v = 7$, $e = 11$, and $r = 6$, which satisfies $r = e - v + 2$, since $11 - 7 + 2 = 6$.

For Γ_3, $v = 7$, $e = 12$, and $r = 7$, which satisfies $r = e - v + 2$, since $12 - 7 + 2 = 7$.

Γ_4 is not planar. For Γ_4, $v = 7$, $e = 15$, so unfortunately, Corollary 1 will not help us here. Also, notice that there is a vertex of degree 4, and there are triangles in the graph, so Corollary 2 and Corollary 3 are also of no help. However, a careful examination of the graph shows that it contains K_5 as a subgraph, and hence cannot be planar.

18. For each description given, either draw a planar graph that meets the description or prove that no planar graph can meet the description given.

(a) A simple graph with 5 vertices and 8 edges.

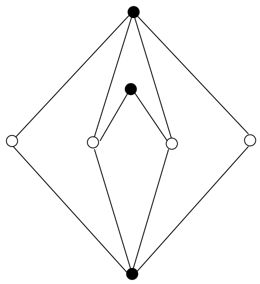


(b) A simple graph with 6 vertices and 13 edges.

Recall that Corollary 1 to Euler's Formula states that if Γ is a connected planar simple graph with e edges and v vertices with $v \geq 3$, then $e \leq 3v - 6$.

Here, $v = 6$ and $e = 13$, so $3v - 6 = 18 - 6 = 12$, so there is no planar graph satisfying this description.

(c) A simple bipartite graph with 7 vertices and 10 edges.



(d) A simple bipartite graph with 7 vertices and 11 edges.

Recall that bipartite graphs have no odd length cycles, and Corollary 3 to Euler's Formula states that if Γ is a connected planar simple graph with e edges and v vertices with $v \geq 3$ and no 3-cycles, then $e \leq 2v - 4$.
Here, $v = 7$ and $e = 11$, so $2v - 4 = 14 - 4 = 10$, so there is no planar graph satisfying this description.

19. Find the value of the following Boolean expressions:

(a) $1 \cdot \overline{(1+0)} + \overline{0}(1+\overline{0})$

$= 1 \cdot \overline{1} + 1(1+1) = 1 \cdot 0 + 1(1) = 0 + 1 = 1$

(b) $\overline{[\overline{1} + (\overline{0} \cdot 1)]} + [\overline{0} + \overline{0} \cdot 1]$

$= \overline{[0 + (1 \cdot 1)]} + [1 + 1 \cdot 1] = \overline{[0+1]} + [1+1] = \overline{1} + 1 = 0 + 1 = 1$

20. Build a value table for the following Boolean functions:

(a) $F(x,y) = x + \overline{x}y$

x	y	$\overline{x}$	$\overline{x}y$	$x+\overline{x}y$
0	0	1	0	0
0	1	1	1	1
1	0	0	0	1
1	1	0	0	1

(b) $F(x,y,z) = xyz + y(\overline{x}+\overline{z})$

x	y	z	xy	xyz	$\overline{x}$	$\overline{z}$	$\overline{x}+\overline{z}$	$y(\overline{x}+\overline{z})$	$xyz+y(\overline{x}+\overline{z})$
0	0	0	0	0	1	1	1	0	0
0	0	1	0	0	1	0	1	0	0
0	1	0	0	0	1	1	1	1	1
0	1	1	0	0	1	0	1	1	1
1	0	0	0	0	0	1	1	0	0
1	0	1	0	0	0	0	0	0	0
1	1	0	1	0	0	1	1	1	1
1	1	1	1	1	0	0	0	0	1

21. Use value tables to determine whether of not the following pairs of Boolean Expressions are equivalent:

(a) $\overline{x}+\overline{y}$ and $\overline{xy}$

x	y	$\overline{x}$	$\overline{y}$	$\overline{x}+\overline{y}$	xy	$\overline{xy}$
0	0	1	1	1	0	1
0	1	1	0	1	0	1
1	0	0	1	1	0	1
1	1	0	0	0	1	0

Since the columns in the table above corresponding to the given expressions are identical, these Boolean expressions are equivalent.

(b) $\overline{xyz}$ and $\overline{x}+\overline{y}+\overline{z}$

x	y	z	xy	xyz	$\overline{xyz}$	$\overline{x}$	$\overline{y}$	$\overline{z}$	$\overline{x}+\overline{y}$	$\overline{x}+\overline{y}+\overline{z}$
0	0	0	0	0	1	1	1	1	1	1
0	0	1	0	0	1	1	1	0	1	1
0	1	0	0	0	1	1	0	1	1	1
0	1	1	0	0	1	1	0	0	1	1
1	0	0	0	0	1	0	1	1	1	1
1	0	1	0	0	1	0	1	0	1	1
1	1	0	1	0	1	0	0	1	0	1
1	1	1	1	1	0	0	0	0	0	0

Since the columns in the table above corresponding to the given expressions are identical, these Boolean expressions are equivalent.

22. Use a 2-column proof to prove each of the following:

(a) $(xyz)+(yz)=yz$

Statement	**Reason**
1. $xyz+yz$	Given
2. $yzx+yz$	Commutative Law
3. $yz(x+1)$	Distributive Law
4. $yz(1)$	Domination Law
5. yz	Identity Law

(b) $\overline{(x+z)\cdot(\overline{y}+z)}=(\overline{x}+y)\cdot\overline{z}$

Statement	**Reason**
1. $\overline{(x+z)\cdot(\overline{y}+z)}$	Given
2. $\overline{(x+z)}+\overline{(\overline{y}+z)}$	De Morgan's Law
3. $\overline{xz}+y\overline{z}$	De Morgan's Law
4. $\overline{zz}+\overline{z}y$	Commutative Law
5. $\overline{z}(\overline{x}+y)$	Distributive Law
6. $(\overline{x}+y)\overline{z}$	Commutative Law

23. Given the following value table:

x	y	z	$F(x,y,z)$	$G(x,y,z)$	$H(x,y,z)$
0	0	0	0	1	0
0	0	1	0	0	1
0	1	0	1	0	1
0	1	1	0	1	0
1	0	0	1	0	1
1	0	1	0	1	0
1	1	0	0	0	0
1	1	1	0	0	1

(a) Find the sum of products expansion for $F(x,y,z)$

$F(x,y,z)=\overline{x}y\overline{z}+x\overline{yz}$

(b) Find the sum of products expansion for $G(x,y,z)$

$G(x,y,z)=\overline{xyz}+\overline{x}yz+x\overline{y}z$

(c) Find the sum of products expansion for $H(x,y,z)$

$H(x,y,z)=\overline{xy}z+\overline{x}y\overline{z}+x\overline{yz}+xyz$

24. Find the sum of products expansion for each of the following Boolean Functions:

(a) $F(x,y,z) = x + \overline{y} + x\overline{z}$

$F(x,y,z) = x + \overline{y} + x\overline{z} = x(y+\overline{y})(z+\overline{z}) + (x+\overline{x})\overline{y}(z+\overline{z}) + x(y+\overline{y})\overline{z}$
$= [xyz + x\overline{y}z + xy\overline{z} + x\overline{y}\,\overline{z}] + [x\overline{y}z + x\overline{y}\,\overline{z} + \overline{x}\,\overline{y}z + \overline{x}\,\overline{y}\,\overline{z}] + [xy\overline{z} + x\overline{y}\,\overline{z}]$
$= xyz + x\overline{y}z + xy\overline{z} + x\overline{y}\,\overline{z} + \overline{x}\,\overline{y}z + \overline{x}\,\overline{y}\,\overline{z}$

(b) $F(x,y,z) = (x+\overline{y})z + x(\overline{y}+z)$

$F(x,y,z) = (x+\overline{y})z + x(\overline{y}+z) = xz + \overline{y}z + x\overline{y} + xz = xz + \overline{y}z + x\overline{y}$
$= x(y+\overline{y})z + (x+\overline{x})\overline{y}z + x\overline{y}(z+\overline{z})$
$= [xyz + x\overline{y}z] + [x\overline{y}z + \overline{x}\,\overline{y}z] + [x\overline{y}z + x\overline{y}\,\overline{z}]$
$= xyz + x\overline{y}z + \overline{x}\,\overline{y}z + x\overline{y}\,\overline{z}$

(c) $F(x,y,z) = xy + \overline{x}y + \overline{y}$

$F(x,y,z) = xy + \overline{x}y + \overline{y} = xy(z+\overline{z}) + \overline{x}y(z+\overline{z}) + (x+\overline{x})\overline{y}(z+\overline{z})$
$= [xyz + xy\overline{z}] + [\overline{x}yz + \overline{x}y\overline{z}] + [x\overline{y}z + \overline{x}\,\overline{y}z + x\overline{y}\,\overline{z} + \overline{x}\,\overline{y}\,\overline{z}]$
$= xyz + xy\overline{z} + \overline{x}yz + \overline{x}y\overline{z} + x\overline{y}z + \overline{x}\,\overline{y}z + x\overline{y}\,\overline{z} + \overline{x}\,\overline{y}\,\overline{z}$

(d) $F(w,x,y,z) = (x+y)(z+\overline{w})$

$F(w,x,y,z) = (x+y)(z+\overline{w}) = xz + yz + \overline{w}x + \overline{w}y = (w+\overline{w})x(y+\overline{y})z + (w+\overline{w})(x+\overline{x})yz + \overline{w}x(y+\overline{y})(z+\overline{z}) +$
$\overline{w}(x+\overline{x})y(z+\overline{z})$
$= [wxyz + \overline{w}xyz + wx\overline{y}z + \overline{w}x\overline{y}z] + [wxyz + \overline{w}xyz + w\overline{x}yz + \overline{w}\,\overline{x}yz] + [\overline{w}xyz + \overline{w}x\overline{y}z + \overline{w}xy\overline{z} + \overline{w}x\overline{y}\,\overline{z}] + [\overline{w}xyz +$
$\overline{w}\,\overline{x}yz + \overline{w}xy\overline{z} + \overline{w}\,\overline{x}y\overline{z}]$
$= wxyz + \overline{w}xyz + wx\overline{y}z + \overline{w}x\overline{y}z + w\overline{x}yz + \overline{w}\,\overline{x}yz + \overline{w}xy\overline{z} + \overline{w}x\overline{y}\,\overline{z} + \overline{w}\,\overline{x}y\overline{z}$

25. Find the sum of products expansion of a Boolean function $F(s,t,x,y,z)$ that has value 1 if and only if an even number of the variables have value 1.

$F(s,t,x,y,z) = st\overline{x}\,\overline{y}\,\overline{z} + s\overline{t}xy\,\overline{z} + s\overline{t}\,\overline{x}y\overline{z} + s\overline{t}\,\overline{x}\,\overline{y}z + \overline{s}txy\,\overline{z} + \overline{s}t\overline{x}y\overline{z} + \overline{s}t\overline{x}\,\overline{y}z + \overline{s}\,\overline{t}xy\overline{z} + \overline{s}\,\overline{t}x\overline{y}z + \overline{s}\,\overline{t}\,\overline{x}yz + stxy\overline{z} + stx\overline{y}z +$
$st\overline{x}yz + s\overline{t}xyz + \overline{s}txyz$

26. Given $F(x,y,z) = x(\overline{y}+z)$

(a) Express $F(x,y,z)$ as a Boolean expression using only the operations $\cdot$ and $^{-}$.

Recall that $u + v = \overline{(\overline{u}\,\overline{v})}$.

Then $F(x,y,z) = x(\overline{y}+z) = x\overline{(y\overline{z})}$

(b) Express $F(x,y,z)$ as a Boolean expression using only the operations $+$ and $^{-}$.

Recall that $uv = \overline{(\overline{u}+\overline{v})}$.

Then $F(x,y,z) = x(\overline{y}+z) = x\overline{y} + xz = \overline{(\overline{x}+y)} + \overline{(\overline{x}+\overline{z})}$

(c) Express $F(x,y,z)$ as a Boolean expression using only the operation $|$.

Recall that $\overline{x} = x|x$ and $xy = (x|y)\,|\,(x|y)$

Then $F(x,y,z) = x(\overline{y}+z) = x\overline{(y\overline{z})} = x\overline{(y(z|z))} = x\overline{[(y|(z|z))\,|\,(y|(z|z))]}$
$= x\,\{[(y|(z|z))\,|\,(y|(z|z))]\,|\,[(y|(z|z))\,|\,(y|(z|z))]\}$
$= \left\{\left[x\middle|\,\{[(y|(z|z))\,|\,(y|(z|z))]\,|\,[(y|(z|z))\,|\,(y|(z|z))]\}\right]\,\middle|\,\left[x\middle|\,\{[(y|(z|z))\,|\,(y|(z|z))]\,|\,[(y|(z|z))\,|\,(y|(z|z))]\}\right]\right\}$

(d) Express $F(x, y, z)$ as a Boolean expression using only the operation $\downarrow$.

Recall that $\overline{x} = x \downarrow x$ and $xy = (x \downarrow x) \downarrow (y \downarrow y)$

Then $F(x, y, z) = x(\overline{y} + z) = x\overline{(y\overline{z})} = x\overline{(y(z \downarrow z))}$
$= x\overline{[(y \downarrow y) \downarrow ((z \downarrow z) \downarrow (z \downarrow z))]}$
$= x\left\{[(y \downarrow y) \downarrow ((z \downarrow z) \downarrow (z \downarrow z))] \downarrow [(y \downarrow y) \downarrow ((z \downarrow z) \downarrow (z \downarrow z))]\right\}$
Let $Q = \left\{[(y \downarrow y) \downarrow ((z \downarrow z) \downarrow (z \downarrow z))] \downarrow [(y \downarrow y) \downarrow ((z \downarrow z) \downarrow (z \downarrow z))]\right\}$.
Then we have:
$F(x, y, z) = (x \downarrow x) \downarrow \left\{Q \downarrow Q\right\}$

27. Using *only* the abstract definition of a Boolean Algebra, prove the following:

(a) Prove that the law of the double complement holds. That it, that $\overline{\overline{x}} = x$ for every element x.

Statement	**Reason**
(1) $\overline{\overline{x}}$	Given
(2) $\overline{\overline{x}} \wedge 1$	Identity Law
(3) $\overline{\overline{x}} \wedge (\overline{x} \vee x)$	Complement Law
(4) $(\overline{\overline{x}} \wedge \overline{x}) \vee (\overline{\overline{x}} \wedge x)$	Distributive Law
(5) $(\overline{\overline{x}} \wedge \overline{x}) \vee 0$	Complement Law
(6) $\overline{\overline{x}} \wedge \overline{x} \vee (x \wedge \overline{x})$	Complement Law
(7) $x \vee (\overline{\overline{x}} \wedge \overline{x})$	Distributive Law
(8) $x \vee 0$	Complement Law
(9) x	Identity Law

(b) Prove that De Morgan's Laws hold. That is, that for all x, y, that $\overline{x \vee y} = \overline{x} \wedge \overline{y}$ and $\overline{x \wedge y} = \overline{x} \vee \overline{y}$

This problem is quite challenging – I will give extra credit if you complete this problem. It is due next Monday.

Here is a bonus proof so you can see another example: Prove the idempotent Law $x = x \wedge x$.

Statement	**Reason**
(1) x	Given
(2) $x \wedge 1$	Identity Law
(3) $x \wedge (x \vee \overline{x})$	Complement Law
(4) $(x \wedge x) \vee (x \wedge \overline{x})$	Distributive Law
(5) $(x \wedge x) \vee 0$	Complement Law
(6) $x \wedge x$	Identity Law