Math 311
Planar Graphs Handout **Name:**______________________________

Recall: A graph is called **planar** if it can be drawn in the plane without any edges crossing (where a crossing of edges is the intersection of the arcs representing them at a point other then a common endpoint). Such a drawing is called a **planar representation** of the graph.

Theorem (Euler's Formula): Let Γ be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of Γ. Then $r = e - v + 2$.

Proof:

Corollary 1: If Γ is a connected planar simple graph with e edges and v vertices, where $v \geq 3$, then $e \leq 3v - 6$.

Corollary 2: If Γ is a connected planar simple graph, then Γ has a vertex of degree not exceeding 5.

Note: The proofs of Corollaries 1 and 2 can be found in your textbook. You should read these proofs and make sure that you understand them. You will be given an opportunity to present these proofs in class later this week.

Corollary 3: If Γ is a connected planar simple graph with e edges and v vertices, with $v \geq 3$ and no circuits of length three, then $e \leq 2v - 4$.

Note: The proof of Corollary 3 is a portfolio eligible proof. You should prove this from first principles, but you may find the structure of the proof of Corollary 1 helpful when writing this proof.

Exercises:

1. Prove that K_5 is not planar.
2. Prove that $K_{3,3}$ is not planar.
3. Prove that $K_{2,n}$ is planar for any n.
4. Prove that $K_{3,n}$ is not planar whenever $n \geq 3$
5. Prove that K_n is not planar whenever $n \geq 5$.

Definition: Two graphs are **homeomorphic** if they can be obtained from the same graph by a sequence of **elementary subdivisions**. Note that in an elementary subdivision, a single edge of a graph is replaced by a path graph of length n.

Kuratowski's Theorem: A graph is non-planar if and only if it contains a subgraph homeomorphic to $K_{3,3}$ or K_5.