Math 311
Relations Activity - Part 2 **Name:**________________________________

Recall:
Let $A = \{0, 1, 2, 3\}$, and consider the following relations on A:

- $R_1 = \{(0, 1), (1, 2), (2, 3), (3, 3)\}$
- $R_2 = \{(0, 0), (0, 1), (1, 1), (1, 2), (2, 2), (2, 3), (3, 3)\}$
- $R_3 = \{(0, 0), (1, 1), (2, 2), (3, 3)\}$
- $R_4 = \{(1, 2), (2, 1), (1, 1)\}$
- $R_5 = \{(0, 0), (0, 1), (1, 0), (1, 1), (2, 2), (1, 3), (3, 1), (3, 3)\}$

New Definition: Given two relations $R \subseteq A \times B$ and $S \subseteq B \times C$, the **composite relation** $S \circ R = \{(a, c) : a \in A, b \in B, c \in C \, \text{suchthat} \, (a, b) \in R \text{and} \, (b, c) \in S\}$.

Examples: $R_2 \circ R_1 = \{(0, 1), (0, 2), (1, 2), (1, 3), (2, 3), (3, 3)\}$ while $R_1 \circ R_1 = R^2 = \{(0, 2), (1, 3), (2, 3), (3, 3)\}$.

Note: If $R \subset A \times A$, we write $R \circ R = R^2$, and, more generally, we write $R^{n+1} = R^n \circ R$ for any $n \in \mathbb{N}$. Moreover, if $R^2 \subseteq R$, then R is transitive.

1. Last time, most of you were unable to correctly identify which of these relations are transitive and which are not. In fact, only R_3 is transitive.

   (a) Find $R_3^2 = R_3 \circ R_3$ directly. Use this result to argue that R_3 is transitive.

   (b) For each of the others, find a specific counterexample that demonstrates that the relation is **not** transitive.

   (c) Use the Boolean product of binary matrices to find $M_{R_3 \circ R_3}$. Compare this matrix to the matrix M_{R_3} and argue that R_3 is transitive.

2. Suppose that R and S are relations on a set B. Further suppose that R and S are both reflexive.

   (a) Is $R \cup S$ reflexive? Justify your answer.

   (b) Is $R \cap S$ reflexive? Justify your answer.

   (c) Is $R - S$ reflexive? Justify your answer.

   (d) Is $R \circ S$ reflexive? Justify your answer.

   (e) Is $\overline{R}$ reflexive? Justify your answer.

3. Suppose that R and S are relations on a set B. Further suppose that R and S are both symmetric.

   (a) Is $R \cup S$ symmetric? Justify your answer.

   (b) Is $R \cap S$ symmetric? Justify your answer.

   (c) Is $R - S$ symmetric? Justify your answer.

   (d) Is $R \circ S$ symmetric? Justify your answer.

   (e) Is $\overline{R}$ symmetric? Justify your answer.