Math 311
Set Proofs Handout and Activity

Recall that a set A is a subset of a set B, written $A \subseteq B$ if every element of the set A is also an element of the set B. To show that one set is a subset of another set using a paragraph proof, we usually use what is called a "general element argument". Here is an example:

Example 1: We will prove that $A \cap B \subset A$

Proof: Let x be an arbitrary element of $A \cap B$. By definition of set intersection, since $x \in A \cap B$, then $x \in A$ and $x \in B$. In particular, $x \in A$. Since every element of $A \cap B$ is also an element of A, $A \cap B \subseteq A$ $\square$

Since that was a fairly straightforward example, let's try another.

Example 2: We will prove that If $A \subseteq B$, then $\overline{B} \subseteq \overline{A}$.

Proof: Let $x \in \overline{B}$. By definition of set complement, $x \notin B$. Recall that since $A \subseteq B$, whenever $y \in A$, we also have $y \in B$. Therefore, using contraposition, whenever $y \notin B$, we must have $y \notin A$. From this, since $x \notin B$, then $x \notin A$. Therefore $x \in \overline{A}$. Hence $\overline{B} \subseteq \overline{A}$. $\square$

Lastly, in order to formally prove that two sets are equal, say $S = T$, we must show that $S \subseteq T$ and that $T \subseteq S$. This will require two general element arguments.

Example 3: We will prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Proof:
"$\subseteq$" Let $x \in A \cup (B \cap C)$. Then $x \in A$ or $x \in B \cap C$. We will consider these two cases separately.

Case 1: Suppose $x \in A$. Then, by definition of set union, $x \in A \cup B$. Similarly, $x \in A \cup C$. Thus. by definition of set intersection, we must have $x \in (A \cup B) \cap (A \cup C)$.

Case 2: Suppose $x \in B \cap C$. Then, by definition of set intersection, $x \in B$ and $x \in C$. Since $x \in B$, then again by the definition of set union, $x \in A \cup B$. Similarly, since $x \in C$, then $x \in A \cup C$. Hence, again by definition of set intersection, $x \in (A \cup B) \cap (A \cup C)$.

Since these are the only possible cases, then $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$

"$\supseteq$" Let $x \in (A \cup B) \cap (A \cup C)$. Then, by definition of set intersection, $x \in A \cup B$ and $x \in A \cup C$. We will once again split into cases.

Case 1: Suppose $x \in A$. Then, by definition of set union, $x \in A \cup (B \cap C)$.

Case 2: Suppose $x \notin A$. Since $x \in A \cup B$, then, by definition of set union, we must have $x \in B$. Similarly, since $x \in A \cup C$, we must have $x \in C$. Therefore, by definition of set intersection, we have $x \in B \cap C$. Hence, again by definition of set union, $A \cup (B \cap C)$.

Since these are the only possible cases, then $A \cup (B \cap C) \supseteq (A \cup B) \cap (A \cup C)$

Thus $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. $\square$

Instructions: Use general element arguments to show that following (Note that only 3, 4, and 5 may be used as portfolio proofs):

1. **Proposition 1:** $B - A \subseteq B$

2. **Proposition 2:** $A - (A - B) \subseteq B$

3. **Proposition 3:** $A - (A - B) = A \cap B$

4. **Proposition 4:** $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

5. **Proposition 5:** $A \times (B \cap C) = (A \times B) \cap (A \times C)$