Math 311
Uniqueness Proofs Solution

Proposition 2: Show that for any irrational number r, there is a unique integer n such that $|r-n| < \frac{1}{2}$.

Proof:

Existence:

Let r be an irrational number. Let $\ell = \lfloor r \rfloor$ and let $m = \lceil r \rceil$. Since ℓ is just r rounded down and m is r rounded up, we must have $\ell + 1 = m$, and $\ell < r < m$. Also notice that since r is irrational, we must have $r \neq \ell + \frac{1}{2}$, otherwise, r would be rational. Hence there are only two cases to check.

Case 1: If $\ell < r < \ell + \frac{1}{2}$, then $|\ell - r| < \frac{1}{2}$, so we may take $n = \ell$.

Case 2: If $\ell + \frac{1}{2} < r < m$, then $|m - r| < \frac{1}{2}$, so we may take $n = m$.

In either case, there is an integer n such that $|n - r| < \frac{1}{2}$.

Uniqueness:

We must show that the n obtained above is unique. Let p be an integer such that $p \neq n$. We will demonstrate that $|p - r| > \frac{1}{2}$.

Case 1: Suppose $n = \lfloor r \rfloor$. Then $n < r < n + \frac{1}{2}$. Let $\epsilon = |n - r|$. If $p = n + 1$, then $|p - r| = 1 - \epsilon > \frac{1}{2}$. If $p > n + 1$, then $n < r < n + 1 < p$, so $|p - r| > 1$. If $p < n$, then $p < n < r < n + 1$, so $|p - r| > 1$.

Case 2: Suppose $n = \lceil r \rceil$. Then $n - \frac{1}{2} < r < n$. Let $\epsilon = |n - r|$. If $p = n - 1$, then $|p - r| = 1 - \epsilon > \frac{1}{2}$. If $p < n - 1$, then $P < n - 1 < r < n$, so $|p - r| > 1$. If $p > n$, then $n - 1 < r < n < p$, so $|p - r| > 1$.

Since we have exhausted all possible cases, n is unique.