## Cubic Splines

**Definition:** Suppose f(x) is a function defined on [a,b]. Given n+1 points
$a = x_0 < x_1 < ... < x_{n-1} < x_n = b$,
a **cubic spline interpolate S** for f(x) is a function that satisfies the following conditions:

(1) S(x) is a piecewise defined cubic polynomial with $S(x) = S_j(x)$, a cubic polynomial, on the interval $[x_j, x_{j+1}]$ for each $j = 0, 1, ..., n$.
(2) $S(x_j) = f(x_j)$ for each $j = 0, 1, ..., n$.
(3) $S_{j+1}(x_{j+1}) = S_j(x_{j+1})$ for each $j = 0, 1, ..., n-2$.
(4) $S'_{j+1}(x_{j+1}) = S'_j(x_{j+1})$ for each $j = 0, 1, ..., n-2$.
(5) $S''_{j+1}(x_{j+1}) = S''_j(x_{j+1})$ for each $j = 0, 1, ..., n-2$.
(6) One of the following two boundary conditions hold:
(a) The "Natural" or "free" boundary condition: $S''(x_0) = S''(x_n) = 0$.
(b) The "Clamped" boundary condition: $S'(x_0) = f'(x_0)$ and $S'(x_n) = f'(x_n)$.

A spline satisfying 1-5 and 6a is called a **natural spline.**
A spline satisfying 1-5 and 6b is called a **clamped spline.**

**Example:** Suppose we have that $f(0) = 2$, $f(1) = 4$, and $f(3) = 5$.

The defintion of a cubic spline give the following equations:

(1) Let $S_0(x) = 2 + b_0 x + c_0 x^2 + d_0 x^3$ and $S_1(x) = 4 + b_1(x-1) + c_1(x-1)^2 + d_1(x-1)^3$.
(2) $S(0) = S_0(0) = f(0) = 2$, $S(1) = S_1(1) = f(1) = 4$, and $S(3) = f(3) = 5$.
(3) $S_0(1) = 4 = S_1(1) = 2 + b_0 + c_0 + d_0$ and $S(3) = 5 = S_1(3) = 4 + 2\,b_1 + 4\,c_1 + 8\,d_1$.
(4) First notice that $S'_0(x) = b_0 + 2\,c_0 x + 3\,d_0 x^2$ and $S'_1(x) = b_1 + 2\,c_1(x-1) + 3\,d_1(x-1)^2$
Then $S'(1) = S_0'(1) = b_0 + 2\,c_0 + 3\,d_0 = b_1 = S_1'(1)$
(5) Notice that $S_0''(x) = 2\,c_0 + 6\,d_0 x$ and $S_1''(x) = 2\,c_1 + 6\,d_1(x-1)$.
Then $S''(1) = S_0''(1) = 2\,c_0 + 6\,d_0 = 2\,c_1 = S_1''(1)$.
(6) We will look at both possible boundary conditions here, but when finding S(x), we need to choose exactly one condition.
(a) $0 = S''(0) = S_0''(0) = 2\,c_0$ and $0 = S''(3) = S_1''(3) = 2\,c_1 + 12\,d_1$.
(b) To apply the clamped boundary condition we actually need two additional values: suppose $f'(0) = 1$ and $f'(3) = -1$.
Then $1 = f'(0) = S'(0) = S_1'(0) = b_0$ and $-1 = f'(3) = S'(3) = S_1'(3) = b_1 + 4\,c_1 + 12\,d_1$.

To find the free spline, we gather the equations from above into the following system:

$4 + 2\,b_1 + 4\,c_1 + 8\,d_1 = 5;$
$2 + b_0 + c_0 + d_0 = 4;$
$b_0 + 2\,c_0 + 3\,d_0 = b_1;$
$2\,c_0 + 6\,d_0 = 2\,c_1;$
$2\,c_0 = 0$
$2\,c_1 + 12\,d_1 = 0$

From the last equation, we see that $c_0 = 0$. Using this and a little algebra, this simplifies the system to give:

$2\,b_1 + 4\,c_1 + 8\,d_1 = 1;$
$b_0 + d_0 = 2;$
$b_0 + 3\,d_0 - b_1 = 0;$
$3\,d_0 - c_1 = 0;$
$c_1 + 6d_1 = 0;$

Notice that this is a system of 5 equations in 5 unknowns. Solving this system using it matrix form gives

$$\begin{bmatrix} 0 & 0 & 2 & 4 & 8 & 1 \\ 1 & 1 & 0 & 0 & 0 & 2 \\ 1 & 3 & -1 & 0 & 0 & 0 \\ 0 & 3 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 6 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 & 4 & 8 & 1 \\ 1 & 1 & 0 & 0 & 0 & 2 \\ 1 & 3 & -1 & 0 & 0 & 0 \\ 0 & 3 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 6 & 0 \end{bmatrix} \xrightarrow{\text{reduced row echelon form}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \frac{9}{4} \\ 0 & 1 & 0 & 0 & 0 & -\frac{1}{4} \\ 0 & 0 & 1 & 0 & 0 & \frac{3}{2} \\ 0 & 0 & 0 & 1 & 0 & -\frac{3}{4} \\ 0 & 0 & 0 & 0 & 1 & \frac{1}{8} \end{bmatrix}$$

Thus we have $b_0 = \frac{9}{4}$, $c_0 = 0$, $d_0 = -\frac{1}{4}$, $b_1 = \frac{3}{2}$, $c_1 = -\frac{3}{4}$, $d_1 = \frac{1}{8}$.

Thus $S_0 := x \to -\frac{1}{4}x^3 + \frac{9}{4}x + 2 = x \to -\frac{1}{4}x^3 + \frac{9}{4}x + 2$ and

$S_1 := x \to 4 + \frac{3}{2}(x-1) - \frac{3}{4}(x-1)^2 + \frac{1}{8}(x-1)^3 = x \to \frac{5}{2} + \frac{3}{2}x - \frac{3}{4}(x-1)^2 + \frac{1}{8}(x-1)^3$

Notice $S_0(0) = 2$, $S_0(1) = 4$, $S_1(1) = 4$, $S_1(3) = 5$ (we could also verify the derivative conditions)

For the clamped spline, we have:

$2\,b_1 + 4\,c_1 + 8\,d_1 = 1;$
$2 + b_0 + c_0 + d_0 = 4;$
$b_0 + 2\,c_0 + 3\,d_0 = b_1;$
$2\,c_0 + 6\,d_0 = 2\,c_1;$
$b_0 = 1$
$b_1 + 4\,c_1 + 12\,d_1 = -1$

Which simplifies to give:

$2\,b_1 + 4\,c_1 + 8\,d_1 = 1;$
$c_0 + d_0 = 1;$
$2\,c_0 + 3\,d_0 - b_1 = -1;$

$c_0 + 3d_0 - c_1 = 0;$
$b_1 + 4\,c_1 + 12\,d_1 = -1$

Again putting this into matrix form, we have:

$$\begin{bmatrix} 0 & 0 & 2 & 4 & 8 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 2 & 3 & -1 & 0 & 0 & -1 \\ 1 & 3 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 4 & 12 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 & 4 & 8 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 2 & 3 & -1 & 0 & 0 & -1 \\ 1 & 3 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 4 & 12 & -1 \end{bmatrix} \xrightarrow{\text{reduced row echelon form}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \frac{23}{12} \\ 0 & 1 & 0 & 0 & 0 & -\frac{11}{12} \\ 0 & 0 & 1 & 0 & 0 & \frac{25}{12} \\ 0 & 0 & 0 & 1 & 0 & -\frac{5}{6} \\ 0 & 0 & 0 & 0 & 1 & \frac{1}{48} \end{bmatrix}$$

Then we have:

$S_0 := x \rightarrow -\frac{11}{12}x^3 + \frac{23}{12}x^2 + x + 2 = x \rightarrow -\frac{11}{12}\,x^3 + \frac{23}{12}\,x^2 + x + 2$ and

$S_1 := x \rightarrow 4 + \frac{25}{12}(x-1) - \frac{5}{6}(x-1)^2 + \frac{1}{48}(x-1)^3 = x \rightarrow \frac{23}{12} + \frac{25}{12}\,x - \frac{5}{6}\,(x-1)^2 + \frac{1}{48}\,(x-1)^3$

$S_0(0) = 2$, $S_0(1) = 4$, $S_1(1) = 4$, $S_1(3) = 5$ (we could also verify the derivative conditions)