Math 450
Final Exam Review Sheet

**Notes:**
• The time for the final exam will be held from 2:00 to 4:00pm on Monday Dec 12th.
• You may use 1 previously prepared full size sheet of paper ($8.5'' \times 11''$) as reference notes for this exam, but no other textbooks, notes, or references are allowed.

**Chapter 1: Review of Calculus, Round-off Errors and Computer Arithmetic, Algorithms and Convergence**

• Know the statements of the Intermediate Value Theorem, Rolle's Theorem, the Mean Value Theorem, and the Extreme Value Theorem.
• Be able to construct the Taylor Series for given function. Also know how to use Taylor's Theorem to compute an upper bound for the error when using a Taylor Polynomial to approximate the value of a function.
• Understand and be able to carry out rounding, chopping, and $k$-digit arithmetic.
• Understand and be able to compute the absolute error and the relative error of an approximation.
• Understand the definition of an algorithm and know the difference between linear and exponential error growth.
• Understand the definition of the rate of convergence of a limit or sequence. Also be able to find the rate of convergence for a given limit or sequence.

**Chapter 2: Methods for Approximating Roots of Functions, Error Analysis, and Accelerating Convergence**

• Know the Algorithm for the Bisection Method, the hypotheses necessary in order to apply this algorithm, be able to apply it to find a root of a function and be able to find an upper bound on the number of iterations needed to find an approximation of a root to within a given error tolerance.
• Know the Algorithm for Fixed Point Iteration, the hypotheses necessary in order to apply this algorithm, and be able to apply it to find a root of a function. Also be able to find an upper bound on the number of iterations necessary in order to find an approximation of a root to within a given error tolerance.
• Know the proof of Theorem 2.3 and be able to apply the error terms for fixed point methods from Corollary 2.4.
• Given a function $f(x)$ that you want to find a root of, be able to find related function(s) to which Fixed Point iteration can be applied (this includes verifying the hypotheses of Theorem 2.3).
• Be able to derive the formula for Newton's Method.
• Know the Algorithms for Newton's Method, the Secant Method, and the Method of False Position. Also know the hypotheses necessary in order to apply these algorithms, and be able to apply them to find a root of a given function.
• Understand the proof that Newton's Method converges.
• Be able to compare the computational effectiveness of these three algorithms when used to approximate a root of a specific function.
• Know the definition of convergence of a sequence $\{p_n\}_{n=0}^{\infty} \to p$ of **order** $\alpha$ with **asymptotic error constant** $\lambda$. Also be able to apply it to find the order of convergence for a specific sequence, or to determine whether or not a given sequence is *linearly convergent* or *quadratically convergent*.
• Know the definition of a **root of multiplicity** $m$ of a function $f(x)$ and be able to find the multiplicity of a root $p$ of a given function, and to express it in the form: $f(x) = (x-p)^m \cdot q(x)$. Also know the connection between a root $p$ of multiplicity $m$ and the value of derivatives of $f$ evaluated at $p$ as stated in Theorem 2.10
• Know and be able to apply the Modified Newton-Raphson Method to approximate a root of a function.
• Know the definition of the **forward difference operator $\Delta$**
• Be able to Apply Aitken's method and Steffensen's Method to accelerate the convergence of Newton's Method.
• Know the statements of the Fundamental Theorem of Algebra, the Remainder Theorem, and the Factor Theorem.
• Know how to evaluate a polynomial function using synthetic division. Also be able to use synthetic division to find roots of a polynomial, to factor a polynomial, and to aid in carrying out Newton's Method.
• Be able to use Horner's method and deflation in order to find the real and complex roots of a polynomial function, and be able to use Muller's Method to find a root of a function.

**Chapter 3: Interpolation and Approximation of Functions**

• Know the definition of the Lagrange interpolating polynomial and be able to find the Lagrange interpolating polynomial to a function given the values of the function at $n+1$ distinct points.
• Know and be able to find the remainder term for a Lagrange interpolating polynomial and be able to use it to find an upper bound on the error of a particular approximation.
• Know how to compute a divided difference table and be able to use it to find the Newton Divided Difference interpolating polynomial for a function given values of the function at $n+1$ distinct points.
• Know how to extend the method for computing Newton Divided Difference interpolating polynomials to cases where we know information about the derivatives of the original function at various points.
• Be able to extend the method for computing Newton Divided Difference interpolating polynomials to cases where the known values of the function are evenly spaced by computing a forward difference table and interpolating using $P_n(s)$.

• Be able to extend the method for computing Newton Divided Difference interpolating polynomials to cases where the known values of the function are evenly spaced by computing a forward difference table and interpolating using $P_n(s)$.
• Know the definition of the Hermite osculating polynomial approximating a given function and its relationship with Lagrange interpolating polynomials.
• Be able to compute the Hermite interpolating polynomial approximating a given function and use it to approximate a function at a given point.
• Be able to find **piecewise linear** and **cubic spline** interpolating functions. Also know the conditions that a cubic spline must satisfy, and be able to find the system of equations that the cubic spline must satisfy for a specific example (for both "free" and "clamped" endpoints).
• Know and be able to apply the error formula for a clamped cubic spline to find an upper bound on the error when using a clamped spline to approximate the value of a given function.

**Chapter 4: Numerical Differentiation and Integration**

• Be able to derive the **forward difference** and **backward difference** approximation formulas and their error terms by differentiating the first Lagrange Polynomial and its error term.
• Know and be able to apply **two point**, **three-point** and **five-point** formulas to approximate the derivative of a function at a given point. Also be able to find an upper bound on the error in these approximations.
• Given a table of values for a given function, be able to select and apply the most accurate method for approximating the derivative for each of the given $x$ values. Also be able to use first order derivative approximations in order to recursively estimate values for higher order derivatives.
• Given an approximation formula and its error series, be able to apply Richardson's Extrapolation in order to find a new approximation formula with higher degree order of convergence. Also be able to find the related error term and be able to use the error term to find an upper bound on the error of your approximation.
• Understand and be able to apply the Midpoint Rule, the Trapezoid Rule, and Simpson's Rule to approximate the value of the definite integral of a function over a given interval. Also know and be able to apply the error terms for these rules to find an upper bound on the error of these approximations.
• Understand the definition of the **degree of accuracy** or **precision** of a numerical integration technique. Also, be able to apply this definition (using appropriately chosen functions) to solve for the coefficients for a numerical integration technique.
• Know the basic formulas and error terms for Composite Simpson's Rule and the Composite Trapezoid Rule. Also be able to apply Composite Simpson's Rule and the Composite Trapezoid Rule to both approximate the value of a given definite integral and to find an upper bound on the error of your approximation.
• Understand the how the formulas for Composite Simpson's Rule and the Composite Trapezoid Rule are derived, including their error terms.
• Be able to use the error formulas for Composite Simpson's Rule and the Composite Trapezoid Rule in order to determine the step size and number of intervals necessary in order to approximate a definite integral to within a given error bound.
• Be able to apply Romberg integration to find accurate numerical approximations of a given definite integral. Also be able to find the error term for a Romberg Integration Method.
• Know the key properties of Legendre Polynomials and the statement and proof of Theorem 4.7
• Given a table of roots and coefficients, be able to use Gaussian Quadrature to approximate definite integrals over $[-1, 1]$.
• Be able to use a change of coordinates to translate an integral over an arbitrary interval $[a, b]$ to one over $[-1, 1]$.

**Chapter 5: Approximating Solutions to Initial Value Problems**
• Know the definition of a **Lipschitz condition** and a **Lipschitz constant** for a function $f$ over a domain $D$ and know the definition of a **convex** set $D$.
• Be able to show that a given function $f$ is Lipschitz over a domain $D$ either directly or using Theorem 5.3
• Understand the definition of a **well-posed** initial value problem (including the definition of a **perturbed problem**) and be able to use Theorem 5.6 to show that a given initial value problem is well posed.
• Understand Euler's method, and be able to derive it from the first order Taylor Polynomial centered at $t_i$ evaluated at $t_{i+1}$.
• Be able to apply Euler's method to a well posed IVP to approximate the value of the unique solution at a particular point, Also be able to use the error term from Theorem 5.9 in order to find an upper bound on the error in using Euler's Method to approximate a well posed IVP as a specific point.
• Know the general form for both the recursive formula for Taylor's Method of order $n$ and the remainder term that can be used to give an upper bound on the local truncation error.
• Be able to use Taylor's Method of order $n$ to approximate a given well posed IVP at the specific value. This includes knowing how to compute higher order derivatives of $f(t, y)$.
• Know the statement of Taylor's Theorem for functions of two variables (Theorem 5.13), and the related definitions of Taylor Polynomials in two variables and their Remainder terms.
• Know both the statement of the Midpoint Method and how it was derived by matching coefficients to the two variable Taylor Polynomial of degree one.
• Be able to apply the Midpoint Method, Modified Euler's, Heun's Method, and Runge-Kutta of Order Four to approximate solutions to a well posed IVP at a given point.