Math 450
Section 5.4: Runge-Kutta Methods

Main Idea: Our goal is to develop methods that approximate solutions to a well posed initial value problem with $O(h^n)$ truncation error but that do not require computing higher order derivatives of $y' = f(t, y)$. One theoretical tool that we will need to accomplish this is Taylor's Theorem for functions of two variables.

Theorem 5.13: Taylor's Theorem Suppose $f(t, y)$ and all of its partial derivatives of order $\leq n+1$ are continuous on $D = \{(t, y)\,|\, a \leq t \leq b,\ c \leq y \leq d\}$ and let $(t_0, y_0) \in D$. For every $(t, y) \in D$, there is a z between t and t_0 and a μ between y and y_0 with $f(t, y) = P_n(t, y) + R_n(t, y)$, where

$$P_n(t,y) = f(t_0,y_0) + \left[(t-t_0)\frac{\partial f}{\partial t}(t_0,y_0) + (y-y_0)\frac{\partial f}{\partial y}(t_0,y_0)\right]$$
$$+\left[\frac{(t-t_0)^2}{2}\frac{\partial^2 f}{\partial t^2}(t_0,y_0) + (t-t_0)(y-y_0)\frac{\partial^2 f}{\partial t\partial y}(t_0,y_0) + \frac{(y-y_0)^2}{2}\frac{\partial^2 f}{\partial y^2}(t_0,y_0)\right]$$
$$+\cdots+\left[\frac{1}{n!}\sum_{j=0}^{n}\begin{pmatrix} n \\ j \end{pmatrix}(t-t_0)^{n-j}(y-y_0)^j\frac{\partial^n f}{\partial t^{n-j}\partial y^j}(t_0,y_0)\right]$$

and $R_n(t,y) = \left[\frac{1}{(n+1)!}\sum_{j=0}^{n+1}\begin{pmatrix} n+1 \\ j \end{pmatrix}(t-t_0)^{n+1-j}(y-y_0)^j\frac{\partial^{n+1} f}{\partial t^{n+1-j}\partial y^j}(z,\mu)\right]$

We call $P_n(t, y)$ the nth degree Taylor Polynomial in two variables for $f(t, y)$ about (y_0, t_0), and $R_n(t, y)$ is the remainder term.

Note: The maple command *mtaylor* can be used to compute Taylor Polynomials in more than one variable.

With this theorem in hand, we can derive new approximation methods. We will begin with a second order method. Recall the second order Taylor Method from last section involved the term:
$T^{(2)}(t, y) = f(t, y) + \dfrac{h}{2}f'(t, y)$.
In order to find our new method (which will not require derivative computations), we need to find parameters a_1, α_1, and β_1 such that $a_1 f(t + \alpha_1, y + \beta_1)$ approximates $T^{(2)}(t, y)$ with a truncation error that is at worst $O(h^2)$.

Notice that, by the chain rule, $f'(t, y) = \frac{df}{dt}(t, y) = \frac{\partial f}{\partial t}(t, y) + \frac{\partial f}{\partial y}(t, y)y'(t)$, where $y'(t) = f(t, y)$.
Thus we have $T^{(2)}(t, y) = f(t, y) + \frac{h}{2}\frac{\partial f}{\partial t}(t, y) + \frac{h}{2}\frac{\partial f}{\partial y}(t, y)f(t, y)$ $(*)$.

We compare this to the first degree Taylor Polynomial in 2-variables for $a_1 f(t + \alpha_1, y + \beta_1)$:

$a_1 f(t+\alpha_1, y+\beta_1) = a_1 f(t, y) + a_1\alpha_1\frac{\partial f}{\partial t}(t, y) + a_1\beta_1\frac{\partial f}{\partial y}(t, y) + a_1 R_1(t+\alpha_1, y+\beta_1)$ $(**)$

Where $R_1(t+\alpha_1, y+\beta_1) = \frac{\alpha_1^2}{2}\frac{\partial^2 f}{\partial t^2}(z,\mu) + \alpha_1\beta_1\frac{\partial^2 f}{\partial t\partial y}(z,\mu) + \frac{\beta_1^2}{2}\frac{\partial^2 f}{\partial y^2}(z,\mu)$.

Comparing coefficients in $(*)$ and $(**)$ above gives: $a_1 = 1$, $a_1\alpha_1 = \frac{h}{2}$, so $\alpha_1 = \frac{h}{2}$, and $a_1\beta_1 = \frac{h}{2}f(t, y)$, so $\beta_1 = \frac{h}{2}f(t, y)$.

Thus $T^{(2)}(t, y) = f\left(t + \frac{h}{2}, y + \frac{h}{2}f(t, y)\right) - R_1\left(t + \frac{h}{2}, y + \frac{h}{2}f(t, y)\right)$.
But $R_1\left(t + \frac{h}{2}, y + \frac{h}{2}f(t, y)\right) = \frac{h^2}{8}\frac{\partial^2 f}{\partial t^2}(z,\mu) + \frac{h^2}{4}f(t, y))\frac{\partial^2 f}{\partial t\partial y}(z,\mu) + \frac{h^2}{8}\left(f(t, y)\right)^2\frac{\partial^2 f}{\partial y^2}(z,\mu)$. Hence, provided that all of the second order partials are bounded, the truncation error for this approximation is $O(h^2)$.

Note: The result of this derivation is the following 2nd order Runge-Kutta approximation method.

The Midpoint Method: Given a well posed initial value problem $y' = f(t, y)$, $y(a) = \alpha$, we use the following procedure:
$w_0 = \alpha$
$w_{i+1} = w_i + hf\left(t_i + \frac{h}{2}, w_i + \frac{h}{2}f(t_i, w_i)\right)$, for each $i = 0, 1, \cdots, N-1$.

Note: In the interest of time, we omit the details of the derivation of higher order Runge-Kutta Methods. By applying the same method of introducing parameters and matching the coefficients of the higher order Taylor Method in one variable to the coefficients of the Taylor Polynomial in two variables evaluated on the input associated with the introduced parameters and matching coefficients, one can develop the following methods.

$O(h^2)$ Methods derived from $T^{(3)}(t, y)$:

Modified Euler's Method: Given a well posed initial value problem $y' = f(t, y)$, $y(a) = \alpha$, we use the following procedure:
$w_0 = \alpha$
$w_{i+1} = w_i + \frac{h}{2}\left[f(t_i, w_i) + f(t_{i+1}, w_i + hf(t_i, w_i))\right]$, for each $i = 0, 1, \cdots, N-1$.

Heun's Method: Given a well posed initial value problem $y' = f(t, y)$, $y(a) = \alpha$, we use the following procedure:
$w_0 = \alpha$
$w_{i+1} = w_i + \frac{h}{4}\left[f(t_i, w_i) + 3f\left(t_i + \frac{2}{3}h, w_i + \frac{2}{3}hf(t_i + \frac{h}{3}, w_i + \frac{h}{3}f(t_i, w_i))\right)\right]$, for each $i = 0, 1, \cdots, N-1$.

The $O(h^4)$ Method derived from $T^{(4)}(t, y)$:

Runge-Kutta Order Four Method: Given a well posed initial value problem $y' = f(t, y)$, $y(a) = \alpha$, we use the following procedure:
$w_0 = \alpha$
$k_1 = hf(t_i, w_i)$
$k_2 = hf\left(t_i + \frac{h}{2}, w_i + \frac{1}{2}k_1\right)$
$k_3 = hf\left(t_i + \frac{h}{2}, w_i + \frac{1}{2}k_2\right)$
$k_4 = hf(t_{i+1}, w_1 + k_3)$
$w_{i+1} = w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$, for each $i = 0, 1, \cdots, N-1$.

Example: Consider the initial value problem $y' = y - t^2 + 1$, $0 \le t \le 2$, $y(0) = 0.5$, we take $h = 1$ (so $N = 2$).
• Using the Midpoint Method, we have:

$w_0 = 0.5.$

$w_1 = 0.5 + 1f\left(0 + \frac{1}{2}, 0.5 + \frac{1}{2}f(0, 0.5)\right) = 2.5$

$w_2 = 2.5 + 1f\left(1 + \frac{1}{2}, 2.5 + \frac{1}{2}f(1, 2.5)\right) = 5.0$

• Using the Modified Euler Method, we have:
$w_0 = 0.5.$

$w_1 = 0.5 + \frac{1}{2}f\left(f(0, 0.5) + f(1, 0.5 + (1)f(0, 0.5))\right) = 2.25$

$w_2 = 2.25 + \frac{1}{2}f\left(f(1, 2.25) + f(2, 2.25 + (1)f(1, 2.25))\right) = 4.125$

• Using Heun's Method, we have:
$w_0 = 0.5.$

$w_1 = 0.5 + \frac{1}{4}f\left[f(0, 0.5) + 3f\left(0 + \frac{2}{3}(1), 0.5 + \frac{2}{3}(1)f(0, 0.5)\right)\right] \approx 2.416666667$

$w_2 = 2.416666667 + \frac{1}{4}f\left[f(1, 2.416666667) + 3f\left(1 + \frac{2}{3}(1), 2.416666667 + \frac{2}{3}(1)f(1, 2.416666667)\right)\right]$
≈ 4.708333333

• Using the Runge-Kutta Order Four Method, we have:
$w_0 = 0.5.$

$k_1 = 1.5$, $k_2 = 2.0$, $k_3 = 2.25$, $k_4 = 2.75$

$w_1 \approx 2.625$

$k_1 = 2.625$, $k_2 = 2.6875$, $k_3 = 2.71875$, $k_4 = 2.34375$

$w_2 \approx 5.255208333$

Note: The actual value of $y(2) \approx 5.3054720$.
Using a smaller h would greatly improve the results of these approximation methods.