Math 476 - Abstract Algebra 1
Day 13 Group Assignment **Name:**____________________________________

Cyclic Groups and the Symmetric Groups

Recall: Definition 22.14 A group G is **cyclic group** id $G = \langle a \rangle$ for some $a \in G$.

Recall: Theorem 23.2 Every subgroup of a cyclic group is cyclic.

Theorem 23.5 Let G be a group with identity e and let a be an element of G of order n. Then:

- $a^n = e$ and, moreover, n is the smallest positive integer to that $a^n = e$.
- If s is an integer so that $a^s = e$, then n divides s.

Theorem 23.6 Let $G = \langle a \rangle$ be a cyclic group of order n, and let $k \in \mathbb{Z}$. Then:

- $\langle a^k \rangle = \langle a^{\gcd(k,n)} \rangle$
- $|\langle a^k \rangle| = \frac{n}{\gcd(k,n)}$

Theorem 23.7 Let G be a finite cyclic group of order n. For each positive divisor m of n, there is exactly one subgroup of order m, and these are the only subgroups of G.

1. The goal of this Activity is to understand the proof of Theorem 23.7. Let $G = \langle a \rangle$ be a finite cyclic group of order n, and let m be a positive divisor of n.

   (a) Use Theorem 23.6 to explain why there is a subgroup of order m. Describe this subgroup as clearly as possible.

   (b) Next, suppose that G contains subgroups H and K of order m.

      i. Explain why we know that $H = \langle a^s \rangle$ and $K = \langle a^t \rangle$ for some integers s and t.

      ii. Explain why the previous fact along with Theorem 23.5 allows us to conclude that $H = K$.

   (c) Describe, in your own words, what this tells us about the subgroups of a finite cyclic groups.

2. Let $G = \mathbb{Z}_{30}$. Find all elements of G that have order 30.

3. Let a be an element of a group with $|a| = 15$. Find the orders of a^3, a^5, a^2, a^6, and a^{10}.

Definition 25.2: A **permutation** of a set S is a bijection $f : S \to S$.

4. Let $\mathcal{P}$ be the set of all permutations from a set S to itself. Determine whether or not $\mathcal{P}$ is a group under the operation function composition.

Permutation Notation and Cycles

Consider the following permutation: $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 5 & 2 & 1 & 4 \end{pmatrix}$

If we look at how this permutation "acts", it sends 1 to 3, 3 to 5, and 5 to 1. This forms the cycle $1 \to 3 \to 5 \to 1$. Similarly, it sends 2 to 6, 6 to 4, and 4 to 2, forming the cycle $2 \to 6 \to 4 \to 2$.

Using this idea, instead of writing permutations using "double row notation" (as shown above), we can use *cycle notation* to represent a permutation. In this example, we would write $\sigma = (135)(264)$. In this notation, σ is represented as consisting of two disjoint cycles, each of length three. In general, a permutation can be a single cycle, or many disjoint cycles, and the cycles could be of different lengths (they can even have length 1).

5. Find the disjoint cycle decomposition representation for each of the following permutations:

(a) $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 5 & 4 & 2 & 1 \end{pmatrix}$ (b) $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 7 & 6 & 1 & 3 & 5 & 2 \end{pmatrix}$ (c) $\omega = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 1 & 6 & 8 & 2 & 7 & 5 \end{pmatrix}$