Math 476 - Abstract Algebra 1
Day 14 Group Assignment **Name:**________________________________

The Symmetric Groups

Recall: Definition 25.2: A **permutation** of a set S is a bijection $f : S \to S$.

Theorem 25.4 Let S be a set and let $P(S)$ denote the collection of permutations of S. Then $P(S)$ is a group under the operation of composition of functions.

Definition 25.5 The **symmetric group** of degree n is the group $S_n = P(\{1, 2, \cdots, n\})$.

Last time, we investigated two different notations for expressing permutations: "two-row" notation and "cycle notation". There is actually more than one way to represent a single permutation in cycle notation. However, permutations do end up having a unique representation (up to cycle rearrangement) if we insist that the cycles are all disjoint.

Definition 25.10 Two cycles $\sigma = (a_1 a_2 \cdots a_k)$ and $\tau = (b_1 b_2 \cdots b_m)$ are **disjoint** if $a_i \neq b_j$ for all $1 \leq i \leq k$ and $1 \leq j \leq m$ [that is, no element appears in both cycles].

Theorem 25.11 Let n be a positive integer. Every permutation in S_n is either a cycle or can be written as a product of disjoint cycles.

Note: This theorem can be proved using strong induction on n. See pp. 337-338 in your textbook.

1. Express each element of S_3 as a product of disjoint cycles.

2. How many permutations are in S_4? How does this compare with the number of elements in D_4 (the symmetries of a square)?

3. How many elements are there in S_n? For which n is S_n abelian?

Note: Another name for a cycle of length 2 is a **transposition**. And cycle can be written as a product of transpositions as follows:

$\sigma = (a_1 a_2 \cdots a_k) = (a_1 a_2)(a_2 a_3)(a_3 a_4) \cdots (a_{k-2} a_{k-1})(a_{k-1} a_k)$

Another important fact to notice is that these transposition representations are not unique. For example, here is another representation of σ:

$\sigma = (a_1 a_2 \cdots a_k) = (a_1 a_k)(a_1 a_{k-1})(a_1 a_{k-2}) \cdots (a_1 a_2)$

4. Let $\tau = (12345)$. Express τ as a product of transpositions in two different ways. Briefly justify why you are confident that both representations are correct.

5. Express the permutation σ as a product of transpositions: $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 1 & 6 & 2 & 3 & 5 \end{pmatrix}$

Definition 25.16 Let $n \in \mathbb{Z}^+$. A permutation $\sigma \in S_n$ is **even** if σ can be written as a product of an even number of transpositions. A permutation σ is **odd** if σ is not even.

Note: In order for this definition to make sense, the following theorem is required (why?).

Theorem 25.14 Let $n \in \mathbb{Z}^+$ and let permutation $\sigma \in S_n$. No matter how σ is written as a product of transpositions, the number of transpositions in the product will have the same parity [That is, the number of transpositions may be different for different representations, but a single permutation can never have both an even and odd representation – see pp. 339-340 in your book for a proof of this theorem].

Definition 25.19 For $n \geq 2$, the **alternating group** A_n is the subgroup of S_n consisting of the even permutations of S_n.

6. Find the parity of the permutations τ and σ from problems 4 and 5 above.

7. Show that A_n as defined above is a subgroup of S_n.

8. Find the elements of A_3. Then find the elements of A_4 [Hint: how does the order of A_n compare to the order of S_n?].