Math 476 - Abstract Algebra 1
Day 15 Group Assignment **Name:**______________________________

Recall: Given integers $a, b \in \mathbb{Z}$, we say that a is congruent to b modulo n if $b - a$ is a multiple of n. We can rephrase this in terms of the subgroup $n\mathbb{Z} = \{nk : k \in \mathbb{Z}\}$ of the group $\mathbb{Z}$ by observing that $a \equiv b(\text{mod} n)$ if $b - a \in n\mathbb{Z}$. With this in mind, we can define a similar relation on any group G relative to a subgroup H. We will try this out in the first problem below.

1. Let G be the group $\mathbb{Z}_8$ (under addition). Let $H = \langle [4] \rangle = \{[0], [4]\}$. Define a relation $\sim_H$ on $\mathbb{Z}_8$ by $a \sim_H b$ if and only if $b - a \in H$ (Note that we are using additive notation).

   (a) Find all elements in $\mathbb{Z}_8$ that are related to $[2]$ via $\sim_H$.

   (b) For each $g \in \mathbb{Z}_8$, find all of the elements that are related to g via $\sim_H$.

   (c) What patterns and properties do you notice about the collection of sets that you found in part (b) above?

Definition 26.3: Let G be a group and H a subgroup of G. Let $\sim_H$ be the relation on G such that for all $a, b \in G$, $a \sim_H b$ if and only if $a^{-1}b \in H$ (Note that we are now using multiplicative notation).

2. Let $G = S_3$ and $H = \langle (12) \rangle$. Find all elements $\sigma \in S_3$ such that $\sigma \sim_H (123)$.

3. Let G be a group with identity e and let H be a subgroup of G.

   (a) Prove that $\sim_H$ is a reflexive relation on G.

   (b) Prove that $\sim_H$ is a symmetric relation on G.

   (c) Prove that $\sim_H$ is a transitive relation on G.

Definition 26.6: Let G be a group and H a subgroup of G. Let $g \in G$. The **left coset** of H in G containing g is the set $gH = \{gh : h \in H\}$. Similarly, the **right coset** of H in G containing g is the set $Hg = \{hg : h \in H\}$. The element g is called the **coset representative** of gH (or Hg).

Note: We are once again using multiplicative notation. In additive notation, we would write $a + H$ or $H = a$ for left and right cosets, respectively. Also notice that the equivalence classes under $\sim_H$ are precisely the same as the left cosets of H in G. Given this, the following Theorem follows directly from facts that we already know about equivalence classes.

Theorem 26.7 Let G be a finite group and H a subgroup of G.

- If a and b are in G, then $aH = bH$ or $aH \cap bH = \emptyset$.
- The group G can be written as a disjoint union of left cosets of H.

4. Let $G = S_3$.

   (a) Let $H = \langle (12) \rangle$ Find the left cosets of H in G.

   (b) Let $H = \langle (12) \rangle$ Find the right cosets of H in G.

   (c) Let $H = \langle (123) \rangle$ Find the left cosets of H in G.

   (d) Let $H = \langle (123) \rangle$ Find the right cosets of H in G.

5. If G is any group, H a subgroup of G, and g an element of G, what, if anything, can we say about how gH and Hg compare to each other?