Math 476 - Abstract Algebra 1
Day 17 Group Assignment **Name:**______________________________

Normal Subgroups and Quotient Groups

Recall: Let G be a group (we will write its operation using multiplicative notation), H a subgroup of G, and $g \in G$. Then the left coset of H in g is the set $gH = \{gh, h \in H\}$.

Let G/H to denote the collection of all distinct left cosets of the subgroup H in the group G. Our goal for this activity is to determine when it is possible to make the set G/H into a group (in a way the corresponds to the original operation in G). In order to do this, we will need to define the operation on G/H. We will do so as follows:

Suppose aH and bH are elements in G/H. Then the product $(aH)(bH) = (ab)H$. Our main goal in the activity that follows is to see if (and when) this operation ends up being well defined.

1. Consider the left cosets of $H = \langle(123)\rangle$ in $G = S_3$ that you found in Monday's daily group work assignment.

   (a) For which $g \in G$ is $gH = H$?

   (b) For which $g \in G$ is $gH = (12)H$?

   (c) Pick a coset g_1H from part (a) and a coset g_2H from part (b) and form the product $(g_1g_2)H$.

   (d) Pick a different coset g_3H from part (a) and a coset g_4H from part (b) and form the product $(g_3g_4)H$. How does the result compare with what you found in part (c)?

   (e) Keep the coset g_3H from part and pick a third coset g_5H from part (b) and form the product $(g_3g_5)H$. How does this compare with your previous results?

   (f) Based on this, comment on what needs to be true in order for this operation to be well defined.

2. Let G be the the group of order 12 given in table 27.4 on p. 362 of your textbook. Let $H = \{I, a_9, a_{10}, a_{11}\}$ and $K = \{I, a_3, a_4\}$. Notice that both H and K are subgroups of G (this can be observed from the table – focus on the rows and columns for just those elements).

   (a) Give a complete listing of all distinct left cosets of H in G (this is G/H).

(b) Give a complete listing of all distinct right cosets of H in G. How do they compare to the left cosets?

(c) Give a complete listing of all distinct left cosets of K in G (this is G/K).

(d) Give a complete listing of all distinct right cosets of K in G. How do they compare to the left cosets?

(e) Show, using a specific example, that the standard product operation on G/K is **not** well-defined.

Definition 27.4: Let G be a group. A subgroup N of G is **normal** in G (or we say it is a **normal subgroup** of G) if $aN = Na$ for all $a \in G$.

(f) Applying this definition, for the group and subgroups defined above, is H normal in G? Is K normal in G? Justify your answers.

Note: The normal subgroups of G are exactly the subgroups for which the multiplication we defined on G/N, the collection of distinct left cosets, ends up being well-defined. Next time, we will look at a more convenient way to check to see if a particular subgroup is normal, and we will use this "product" operation on cosets to define a new group.