Math 476 - Abstract Algebra 1
Day 18 Group Assignment **Name:**________________________________

Normal Subgroups Quotient Groups

Recall – Definition 27.4: Let G be a group. A subgroup N of G is **normal** in G (or we say it is a **normal subgroup** of G) if $aN = Na$ for all $a \in G$.

Notation: If S is a subset of a group G, we write $S \subseteq G$. If H is a subgroup of G, we write $H < G$. If N is a normal subgroup of G we write $N \lhd G$. The following theorem is useful for showing that a particular subgroup is normal in G.

Theorem 27.5: Let G be a group and N a subgroup of G. Then N is normal in G if and only if $aNa^{-1} \subseteq N$ for all $a \in G$, where $aNa^{-1} = \{ana^{-1} : n \in N\}$.

Proof:
The forward implication for this theorem will be one of your homework exercises this week. To prove the reverse implication, suppose that $aNa^{-1} \subseteq N$ for all $a \in G$. We must show that $N \lhd G$. That is, that $aN = Na$ for all $a \in G$. We will show set equality by showing containment in both directions.

Suppose that $x \in aN$. Then $x = an$ for some $n \in N$. Since $aNa^{-1} \subseteq N$, then $ana^{-1} \in N$. Then, for some $n' \in N$, we have $n' = ana^{-1}$. Then, right multiplying by a, we have $n'a = an = x$. Thus $x \in Na$. This shows that $aN \subseteq Na$.

To see the reverse containment, suppose $y \in Na$. Then $y = na$ for some $n \in N$. As before, since $aNa^{-1} \subseteq N$, this time replacing a with a^{-1}, we see $a^{-1}na = n''$ for some $n'' \in N$. Then, right multiplying by a gives $y = na = an''$ so $y \in aN$. This shows that $Na \subseteq aN$. Thus $aN = Na$.

1. Use Theorem 27.5 to determine whether or not the given subgroup N is a normal subgroup of the group G.

   (a) $G = \mathbb{Z}$, $N = \langle 5 \rangle$.

   (b) $G = S_3$, $N = \langle (123) \rangle$.

   Definition 27.8: Let G be a group and N a normal subgroup of G. The **quotient group** (or **factor group**) of G by N is the group $G/N = \{aN : a \in G\}$ with the operation $(aN)(bN) = (ab)N$ for all $a, b \in G$.

2. For each group G and normal subgroup N below, construct an operation table for the quotient group G/N.

   (a) $G = \mathbb{Z}_8$, $N = \langle [2] \rangle$.

(b) $G = D_4$, $N = \langle R^2 \rangle$.

3. Let G be an abelian group and let H be a subgroup of G. Must H be normal in G? Justify your answer.

Theorem 27.10 If G is a group and $G/Z(G)$ is cyclic, then G is abelian.

Note: Be sure to take time to read the proof of this theorem on pp. 364-365 in your textbook. We will discuss it in more detail in class later this week.

Theorem 27.11 (Cauchy's Theorem for Finite Abelian Groups). Let G be an Abelian group of finite order n. If p is a prime divisor of n, then G contains an element of order p.

The proof of this theorem involves a fairly subtle induction argument. See pp. 366-367 in your textbook.

Corollary 27.13 Any Abelian group of order pq, where p and q are distinct primes, is cyclic.

Proof Sketch:

Let G be an Abelian group of order pq, where p and q are distinct primes. By Cauchy's Theorem for Finite Abelian Groups, G has an element a of order p and an element b of order q. Claim: $|ab| = pq$, so $G = \langle ab \rangle$ (and hence G is cyclic).

4. **Extra Credit:** Let G be a group with $a, b \in G$. If $|a| = p$, $|b| = q$ with $\gcd(p, q) = 1$ and $ab = ba$, show that $|ab| = pq$.

Definition 27.14 A group G is **simple** if G has no non-trivial normal proper subgroups.

5. **Claim:** A_5 is simple. State what you would need to show to prove this claim (you don't need to actually show it).