Math 476 - Abstract Algebra 1
Day 20 Group Assignment **Name:**________________________________

Isomorphisms and Invariants

1. Build a multiplication table (also called a Cayley table) for the group $\mathbb{Z}_8$.

Claim: both G_2 and G_3 from the Day 19 group work assignment are isomorphic to $\mathbb{Z}_8$ (and hence to each other). To see that G_2 is isomorphic to $\mathbb{Z}_8$, consider the map φ defined as follows: $\varphi([0]) = \alpha$, $\varphi([1]) = \beta$, $\varphi([2]) = \gamma$, $\varphi([3]) = \delta$, $\varphi([4]) = \epsilon$, $\varphi([5]) = \zeta$, $\varphi([6]) = \eta$, and $\varphi([7]) = \theta$.

Finding an isomorphism between G_3 and $\mathbb{Z}_8$ is a bit more difficult. Notice using a similar strategy of mapping [0] to s, [1] to t, etc. does not yield the same Cayley Table. We claim that the following map is an isomorphism between $\mathbb{Z}_8$ and G_3: $\phi([0]) = s$, $\phi([1]) = w$, $\phi([2]) = t$, $\phi([3]) = x$, $\phi([4]) = u$, $\phi([5]) = y$, $\phi([6]) = v$, and $\phi([7]) = z$.

2. Rewrite the Cayley table for G_3 with rows and columns arranged in the order s, w, t, x, u, y, v, z. Compare this to the operation table you built in problem 1. Is this enough to show that $\mathbb{Z}_8$ and G_3 are isomorphic?

3. Prove that G_1 and G_4 are not isomorphic to $\mathbb{Z}_8$ and are not isomorphic to each other.

For the remainder of this assignment, you are asked to work on proving the following properties of isomorphisms in your groups. Once you feel you have a good argument for a property, a member of your group can volunteer to present your argument to the rest of the class.

4. Let $\varphi : G \to H$ be a group isomorphism. Prove the following properties:
   (a) Prove that $\varphi(e_G) = e_H$.
   (b) Prove that $\varphi(a^{-1}) = (\varphi(a))^{-1}$
   (c) Prove that $\varphi(a^k) = (\varphi(a))^k$ for all positive integers k.
   (d) Prove that for any $a \in G$, $|a| = |\varphi(a)|$.
   (e) Given that there is an isomorphism φ between G and H, that G is cyclic if and only if H is cyclic.
   (f) Given that there is an isomorphism φ between G and H, that G is Abelian if and only if H is Abelian.
   (g) Prove that if K is a subgroup of G, then $\varphi(K)$ is a subgroup of H.
   (h) Given that there is an isomorphism φ between G and H, that G is simple if and only if H is simple.
   (i) If there is an isomorphism φ between G and H, we say G and H are isomorphic, denoted $G \cong H$. Show that $\cong$ is an equivalence relation on the the set of all groups.