Math 476 - Abstract Algebra 1
Day 21 Group Assignment **Name:**________________________________

Isomorphisms and Homomorphisms

Definition 29.11: Let S and T be sets. A mapping $f : S \to T$ is **well-defined** is $f(a) = f(b)$ whenever $a = b$ in S.

Recall: If there is an isomorphism φ between G and H, we say G and H are isomorphic, denoted $G \cong H$.

Theorem 29.14 The relation of group isomorphism ($\cong$) is an equivalence relation on the the set of all groups.

In order to prove this theorem, we must show that this relation is reflexive, symmetric, and transitive. I will leave the proofs of reflexivity and transitivity as presentation problems. Here is an outline of the proof that this relation is symmetric:

Suppose that G and H are groups and that $\varphi : G \to H$ is an isomorphism from G to H. Since φ is a isomorphism, it is a bijection, so there is a well defined inverse map φ^{-1} from H back to G defined via $\varphi^{-1}(y) = x$. whenever $\varphi(x) = y$. To see that this map is well defined, note that if $\varphi^{-1}(y) = x$ and $\varphi^{-1}(y) = x'$ for some $x, x' \in G$, then $y = \varphi(x)$ and $y = \varphi(x')$. Since φ is 1-1, we must have $x = x'$, hence φ^{-1} is well defined.

To see that φ^{-1} is a bijection, note that if $\varphi^{-1}(y) = \varphi^{-1}(y')$ for $y, y' \in H$, then $\exists x \in G$ so that $x = \varphi^{-1}(y) = \varphi^{-1}(y')$, so $\varphi(x) = y = y'$. Since φ is a well defined function, $y = y'$, so φ^{-1} is 1-1. To see φ^{-1} is onto, note that for every $x \in G$, there is some $y \in H$ such that $y = \varphi(x)$. Thus $\varphi^{-1}(y) = x$.

The last detail that remains is to show that φ^{-1} preserves the group operation. Let $y, y' \in H$. Let $x = \varphi^{-1}(y)$, and $x' = \varphi^{-1}(y')$. Then $\varphi(x) = y$, and $\varphi(x') = y'$. Since φ is an isomorphism, $\phi(xx') = \varphi(x)\varphi(x') = yy'$. Thus $\varphi^{-1}(yy') = xx' = \varphi^{-1}(y)\varphi^{-1}(y')$. Therefore, $\varphi^{-1} : H \to G$ is an isomorphism. Hence $\cong$ is a symmetric relation.

Theorem 29.16 Let p be an odd prime and G a non-Abelian group of order $2p$. Then $G \cong D_p$. [We will not go over the proof of this theorem. if you are interested, you should read I-24 and pp. 406-407 in your textbook].

Theorem 29.18 Any finite cyclic group of order n is isomorphic to $\mathbb{Z}_n$.

Proof Sketch: Let a be a generator for a cyclic group G of order n. Define a map $\varphi : G \to \mathbb{Z}_n$ by taking $\varphi(a^k) = [k]$. In particular $\varphi(a) = [1]$, and $\varphi(a^k) = [0]$. We claim that this map is an isomorphism.

Theorem 29.19 Any countably infinite cyclic group is isomorphic to $\mathbb{Z}$. [You will be asked to prove this on your next homework assignment].

Theorem 29.24 (Cayley's Theorem) Every group is a subgroup of a group of permutations. [we will discuss this theorem in more detail later.]

Corollary 29.25 If G is a finite group of order n, then G is isomorphic to a subgroup of the symmetric group S_n.

Definition 30.2 Let G and H be groups. A function φ from G to H is a **homomorphism** of groups if $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in G$.

1. How does this definition differ from that of an isomorphism of groups? is every isomorphism and homomorphism? Is every homomorphism an isomorphism?

   Notation: Here is some vocabulary that is used to distinguish different types of homomorphisms:

   - a **monomorphism** is an injective homomorphism (a homomorphism that is 1-1).
   - an **epimorphism** is an surjective homomorphism (a homomorphism that is onto).
   - an **isomorphism** is a homomorphism that is bijective (a homomorphism that is both 1-1 and onto).

   Note: If $\varphi : G \to H$ is an epimorphism, we call H the **homomorphic image** of G.

2. Why do you think that we require the map to be an epimorphism in order to consider H to be the homomorphic image of G?

Theorem 30.4 Let G and H be groups with identities e_G and e_H, respectively, and let $\varphi : G \to H$ a homomorphism. Then:

(a) $\varphi(e_G) = e_H$.

(b) If $a \in G$, then $\varphi(a^{-1}) = (\varphi(a))^{-1}$

Note: The proofs of these follow closely from the similar results we proved for isomorphisms. Can you adapt those proofs to this situation?

Definition 30.5 Let $\varphi : G \to H$ be a homomorphism of groups and let e_H be the identity element in H. The **kernel** of φ is the set $Ker(\varphi) = \{a \in G : \varphi(a) = e_H\}$.

3. For each of the following, determine whether or not the given map is a group homomorphism. For those that are, find the kernel.

   (a) $G = \mathbb{Z}$, $H = \mathbb{Z}_5$, and $\varphi(k) = [k]_5$.

   (b) $G = \mathbb{Z}_3$, $H = \mathbb{Z}_{18}$, and $\varphi([k]_3) = [6k]_{18}$.

   (c) $G = U_{12}$, $H = \mathbb{Z}_6$, and $\varphi([k]_{12}) = [k]_6$.