Math 476 - Abstract Algebra 1
Day 22 Group Assignment **Name:**________________________________

Group Homomorphisms

Recall: Definition 30.2 Let G and H be groups. A function φ from G to H is a **homomorphism** of groups if $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in G$.

Definition 30.5 Let $\varphi : G \to H$ be a homomorphism of groups and let e_H be the identity element in H. The **kernel** of φ is the set $Ker(\varphi) = \{a \in G : \varphi(a) = e_H\}$.

1. Let G and H be groups with identities e_G and e_H, respectively, and let $\varphi : G \to H$ be a group homomorphism.

   (a) Is e_G in $Ker\ \varphi$? Justify your answer.

   (b) Is $Ker\ \varphi$ closed under the operation in G? Prove your answer.

   (c) If $a \in Ker\ \varphi$, is $a^{-1} \in Ker\ \varphi$? Prove your answer.

   (d) Based on what you showed in (a), (b), and (c) above, is $Ker\ \varphi$ a subgroup? What group is it a subgroup of?

Theorem 30.8: Let G and H be groups, and let $\varphi : G \to H$ be group homomorphism. Then $Ker\ \varphi$ is a normal subgroup of G.

Proof:

Let $K = Ker\varphi$. Note that the work you did in problem 1 (hopefully) shows that $Ker\varphi$ is a subgroup. It remains to show that it is normal. We will do so by demonstrating that $aKa \subseteq K$ for all $a \in G$. Let $a \in G$ and consider $\varphi(aka^{-1})$. Since φ is a homomorphism, $\varphi(aka^{-1}) = \varphi(a)\varphi(k)\varphi(a^{-1})$. Since $k \in K$, we must have $\varphi(k) = e_H$, so $\varphi(aka^{-1}) = \varphi(a)\varphi(k)\varphi(a^{-1}) = \varphi(a)e_H\varphi(a^{-1}) = \varphi(a)\varphi(a^{-1})$. Finally, since $\varphi(a^{-1}) = (\varphi(a))^{-1}$, then $\varphi(a)\varphi(a^{-1}) = \varphi(a)(\varphi(a))^{-1} = e_H$. Hence $aka^{-1} \in K$. Therefore, $aKa \subseteq K$ for all $a \in G$. Thus, $K \triangleleft G$. $\square$

Definition 30.9: Let $\varphi : G \to H$ be a group homomorphism. The **image** of φ is the set $Im(\varphi) = \{\varphi(a) : a \in G\}$.

2. Find the image of the map $\varphi : \mathbb{Z}_9 \to \mathbb{Z}_6$ defined by $\varphi([k]_9) = [2k]_6$ (you do not have to check to see if this map is a homomorphism).

3. Let $\varphi : G \to H$ be a group homomorphism. Determine whether or not $Im(\varphi)$ is a subgroup of H.

4. Let $G = \mathbb{Z}_{24}$, $H = \mathbb{Z}_8$ and let $\varphi : G \to H$ be defined by $\varphi([m]_{24}) = [6m]_8$.

(a) Show that φ is well-defined.

(b) Show that φ is a homomorphism of groups.

(c) Find $Ker\ \varphi$.

(d) Determine the elements of the group G/K. Is G/K Abelian? Is G/K cyclic? Justify your answers.

(e) Find the elements of $R = Im(\varphi)$.

(f) How do the groups G/K and $R = Im(\varphi)$ compare?