 Name:____

Definition 31.2 A p-**group**, where p is a prime, is a group whose order is a power of p.

Note: Our goal, in this section, is to classify all Abelian groups that have finite order. As we will see, p-groups will end up being the building blocks we use to build any finite order Abelian group. We will "combine" these building blocks using the direct sum operation ($\oplus$) that we explored in the portions of I-28 that we discussed previously.

Theorem 31.3 Let p be a prime, and let G be a finite Abelian p-group. Let $a \in G$ be an element of maximal order in G. There there exists a subgroup K of G so that $G = \langle a \rangle \times K$.

To help illustrate the consequences of this theorem, consider the following example:

1. Let $G = U_{60}$. To save us some time, please note that elements of this group along with the order of each element is given on page 436 in your textbook.

   (a) Find an element a of maximal order in U_{60} and compute $\langle a \rangle$.

   (b) Find a maximal subgroup K of G so that $\langle a \rangle \cap K = \{[1]\}$. (Hint: if we want $\langle a \rangle \times K = G$, what must the order of K be? Also note that $[7]^2 = [13]^2 = [17]^2 = [23]^2 = [37]^2 = [43]^2 = [47]^2 = [53]^2 = [49] \in U_{60}$).

   (c) Verify that every element of U_{60} can be written in the form $a^m k$ using the choices of a and K you found in parts (a) and (b) above.

Corollary 31.6: Let p be prime and G a finite Abelian p-group. Then G is an internal direct product of cyclic p-groups.

Example: Let $n = 27$. Determine, up to isomorphism, all Abelian groups of order $n = 27$.

Note that $27 = 3^3$. Then there are three possible ways of decomposing this into factors: 3^3, 3^2:3, and 3:3:3. (this of these as the integer partitions of 3: 3, $2-1$, and $1-1-1$. Note that we do not consider $1-2$ and $2-1$ as different partitions here (Why?).

Then the isomorphism classes are: $\mathbb{Z}_{27}$, $\mathbb{Z}_9 \oplus \mathbb{Z}_3$, and $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$.

Definition 31.10: Let G be a finite Abelian group of order n, and let p be a prime factor of n. The p-**primary component** of G is the set $G(p) = \{a \in G : |a| = p^i \text{ for some non-negative integer } t\}$. That is, $G(p)$ is a set of all elements in G whose order is a power of p.

2. Let G be a finite Abelian group with identity e and let p be a prime divisor of $|G|$. Prove that $G(p)$, as defined above, is a subgroup of G.

Lemma 31.12 If p is a prime factor of $|G|$, then $G(p)$ is a subgroup of G whose order if a power of p.

Proof: See p. 440 in your textbook.

Theorem 31.13 (The Fundamental Theorem of Finite Abelian Groups) Let G be a finite Abelian group. Then $G = G(p_1) \times G(p_2) \times \cdots \times G(p_k)$, there $p_1, p_2, \cdots, p_k$ are the distinct prime factors of $|G|$.

Proof: See pp. 440-441 in your textbook.

Alternative Version of the Fundamental Theorem of Finite Abelian Groups: Let G be a finite Abelian group. Then G is isomorphic to a direct product of cyclic groups: $\mathbb{Z}_{c_1} \oplus \mathbb{Z}_{c_2} \oplus \cdots \oplus \mathbb{Z}_{c_t}$, where $c_i \geq 2$. The integers $c_1, c_2, \cdots, c_t$ are called the **invariant factors** of G.

3. Let G be a finite Abelian group with order $n = |G| = 1800$.

   (a) Find the prime factorization of $n = 1800$ and use it to express G as the direct product of p-groups.

   (b) For each p-group $G(p_i)$ you found in part (a), list all possible ways that $G(p_i)$ could be expressed as a direct product of cyclic groups, up to isomorphism.

   (c) Use the possible forms for each $G(p_i)$ you found in part (b) to create a complete list of isomorphism classes for Abelian groups of order 1800.