Math 476 - Abstract Algebra 1
Day 25 Group Assignment **Name:**______________________________

Definition 6.2 A **rational number** is a real number that can be expressed as the quotient of two integers a and b with $b \neq 0$. The set of all rational, denoted by $\mathbb{Q}$ is defined to be $\mathbb{Q} = \{\frac{a}{b} : a, b \in \mathbb{Z}, and b \neq q\}$.
Note: Two rational numbers $\dfrac{a}{b}$ and $\dfrac{c}{d}$ are considered to be equal if and only if $ad = bc$. Addition and multiplication within $\mathbb{Q}$ are defined as follows:

- $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{ad + bc}{bd}$ and
- $\dfrac{a}{b} \cdot \dfrac{c}{d} = \dfrac{ac}{bd}$ for all $\dfrac{a}{b}, \dfrac{c}{d} \in \mathbb{Q}$.

1. In this exercise, we will investigate some important algebraic properties of $\mathbb{Q}$ under its two operations: addition and multiplication.

   (a) Does $\mathbb{Q}$ have an additive identity? If so, what is it? Does it contain a multiplicative identity? If so, what is it?

   (b) Does every element of $\mathbb{Q}$ have an additive inverse? Justify your answer.

   (c) Which elements of $\mathbb{Q}$ are units (that is, which elements of $\mathbb{Q}$ have a multiplicative inverse)? Are there elements of $\mathbb{Q}$ that do not have a multiplicative inverse in $\mathbb{Q}$?

   (d) Does $x \cdot 0 = 0$ for all $x \in \mathbb{Q}$? (This property is called the *zero property of multiplication*.)

   (e) Does $\mathbb{Q}$ contain any zero divisors? That is, do there exist non-zero elements $x, y \in \mathbb{Q}$ such that $xy = 0$?

   (f) Does additive cancellation hold in $\mathbb{Q}$? That is, if $x + z = y + z$ for some $x, y, x \in \mathbb{Q}$, does it follow that $x = y$?

   (g) Does multiplicative cancellation hold in $\mathbb{Q}$? That is, if $x \cdot z = y \cdot z$ for some $x, y, x \in \mathbb{Q}$, does it follow that $x = y$? Would your answer change is we added the assumption that $z \neq 0$?

   (h) Does $\mathbb{Q}$ satisfy the order axioms on p. 8 of your textbook?

Definition 6.4: A **complex number** is any number of the form $a + bi$. where $a, b \in \mathbb{R}$ and i is an imaginary number with the property that $i^2 = -1$. The real number a is called the **real part** of the complex number $x = a + bi$, and the real number b is called the **imaginary part** of x.

Note: Two complex numbers $a + bi$ and $c + di$ are considered to be equal if and only if $a = c$ and $b = d$. Addition and multiplication within $\mathbb{C}$ are defined as follows:

- $(a + bi) + (c + di) = (a + c) + (b + d)i$ and
- $(a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i$ for all $a + bi,\ c + di \in \mathbb{C}$.

2. In this exercise, we will investigate a few of the algebraic properties of $\mathbb{C}$ under its two operations: addition and multiplication.

   (a) Show that $1 + 0i$ is the multiplicative identity in $\mathbb{C}$. What is the additive identity?

   (b) Find the additive inverse of the complex number $3 + 5i$. Then find an expression for the additive inverse of a general complex number $a + bi$.

   (c) Find the multiplicative inverse of the complex number $3+5i$. Then find an expression for the multiplicative inverse of a general complex number $a + bi$ (assuming that a and b are not both zero).

3. Consider $\mathcal{M}_{2\times 2}(\mathbb{R})$, the set of all 2×2 matrices with real entries.

   (a) Describe how addition is defined in $\mathcal{M}_{2\times 2}(\mathbb{R})$.

   (b) Describe how multiplication is defined in $\mathcal{M}_{2\times 2}(\mathbb{R})$.

   (c) Does $\mathcal{M}_{2\times 2}(\mathbb{R})$ have an additive identity? Does $\mathcal{M}_{2\times 2}(\mathbb{R})$ have a multiplicative identity?

   (d) Does every element of $\mathcal{M}_{2\times 2}(\mathbb{R})$ have an additive inverse? If so, what form does it have?

   (e) Does every element of $\mathcal{M}_{2\times 2}(\mathbb{R})$ have a multiplicative inverse? If so, what form does it have?