Math 476 - Abstract Algebra 1
Day 26 Group Assignment **Name:**______________________________

Definition 6.6 Let S be a set. The **power set** of S, denoted $\mathcal{P}(S)$ is the collection of all subsets of S. That is, $\mathcal{P}(S) = \{T : T \subseteq S\}$. **Example:** If $S = \{a, b\}$, then $\mathcal{P}(S) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$.

For any sets A and B, the **symmetric difference** of A and B, denoted $A\triangle B$, is the set of all elements that belong to either A or B, but not both. That is, $A\triangle B = \{x : a \in A \cup B \text{ and } x \notin A \cap B\}$.

For any natural number n, the number system $\mathcal{P}_n$ is the power set of the set $\{1, 2, \cdots, n\}$, with addition defined as symmetric difference, and multiplication defined as intersection. In other words, $\mathcal{P}_n = \mathcal{P}\{1, 2, \cdots, n\}$, with $A + B = A\triangle B$ and $A \cdot B = A \cap B$.

1. Consider the number system $\mathcal{P}_n$

   (a) Prove that $\triangle$ is associative (See p. 71 in your textbook).

   (b) Build both the "addition" and "multiplication" tables for $\mathcal{P}_3$.

   (c) Does $\mathcal{P}_n$ have an additive identity? If so, what is it? Does it contain a multiplicative identity? If so, what is it?

   (d) Which elements have an additive inverse? Which elements have a multiplicative inverse? Justify your answers.

   (e) Does $\mathcal{P}_n$ contain any zero divisors? Justify your answer.

   (f) Does $\mathcal{P}_n$ satisfy the order axioms on p. 8 of your textbook?

Definition 7.2: A **ring** is a set R together with two binary operations, called addition $(+)$ and multiplication $(\cdot)$, such that all of the following axioms hold:

The Ring Axioms

- The set R **is closed under addition and multiplication**, meaning that for all $x, y \in R$, $x+y] \in R$ and $x \cdot y \in R$.
- **Addition is associative**, meaning that for all $x, y, z \in R$, $(x+y)+z = x+(y+z)$.
- **Addition is commutative**, meaning that for all $x, y \in R$, $x+y = y+x$.
- **The set R contains an additive identity**, also called a **zero element**, meaning that there exists some element $0_R \in R$ such that $x + 0_R = x$ for all $x \in R$.
- **Every element in R has an additive inverse within** R, meaning that for every $x \in R$, there exists $y \in R$ such that $x + y = 0_R$.
- **Multiplication is associative**, meaning that for all $x, y, z \in R$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.
- **Multiplication distributes over addition**, meaning that for all $x, y, z \in R$, $x \cdot (y+z) = x \cdot y + x \cdot z$ and $(x+y) \cdot z = x \cdot z + y \cdot z$.

Note: Although these are our only Axioms for a Ring, there are some other desirable properties that we would like a ring to have. Most of these can be proven to be a consequence of the axioms we already have. Here are two such properties:

Theorem 7.3 Let R be a ring. For all $x, y, z \in R$, if $x + z = y + z$, then $x = y$.

Theorem 7.5 Let R be a ring. Then $0x = 0 = x0$ for all $x \in R$.

2. Use the Ring Axioms to prove Theorem 7.3

3. Use the Ring Axioms to prove Theorem 7.5

Definition 7.7 Let R be a ring. Then R is said to be **commutative** if multiplication in R is commutative – that is, if $x \cdot y = y \cdot x$ for all $x, y \in R$

4. Give an example of a ring R_1 that is commutative and a ring R_2 that is **not** commutative.

Definition 7.8 Let R be a ring. An **identity** for R is an element $1_R \in R$ such that $1_R \neq 0$ and $1_R \cdot x = x = x \cdot 1_R$ for all $x \in R$. If such an element exists, then R is said to be a **ring with identity**.

Note: See p. 82 in your textbook for a list of examples of each type of ring that we have discussed.