Math 476 - Abstract Algebra 1
Day 28 Group Assignment **Name:**______________________________

Integer Multiples and Exponents

Definition 8.2 Let R be a ring, let $n \geq 3$ be an integer, an let $x_1, x_2, \cdots, x_n \in R$. Then we define $x_1 + x_2 + \cdots + x_n = (x_1 + x_2 + \cdots + x_{n-1}) + x_n$ and $x_1 x_2 \cdots x_n = (x_1 x_2 \cdots x_{n-1}) x_n$.

Note: Definition 8.2 is an example of a *recursive definition*. That is, it defines the next instance in terms of one or more previous instances. To complete a particular computation, one would need to apply the recursive definition multiple times. For example, to compute $x_1 + x_2 + x_3 + x_4$, we could apply the definition as follows:

$x_1 + x_2 + x_3 + x_4 = (x_1 + x_2 + x_3) + x_4$; $x_1 + x_2 + x_3 = (x_1 + x_2) + x_3$; compute $x_1 + x_2$; use the result to compute $(x_1 + x_2) + x_3 = x_1 + x_2 + x_3$. Then use this second result to compute $(x_1 + x_2 + x_3) + x_4 = x_1 + x_2 + x_3 + x_4$.

Definition 8.3 Let R be a ring, and let $x \in R$. The the expression $1x$ and x^1 are both defined to be equal to x; that is, $1x = x$ and $x^1 = x$.

Furthermore, for every integer $n \geq 2$, we define the expressions nx and x^n recursively as follows:

- $nx = \underbrace{x + x + \cdots + x}_{n \text{ terms}} = \underbrace{x + x + \cdots + x}_{n-1 \text{ terms}} + x = (n-1)x + x$.
- $x^n = \underbrace{x \cdot x \cdots x}_{n \text{ factors}} = \underbrace{(x \cdot x \cdots x)}_{n-1 \text{ factors}} x = x^{n-1} x$.

Note: We would like to extend these ideas to subtraction, to negative exponents, and to be able to use zero as an exponent. As we will soon see, some of these notions will work for arbitrary elements in any ring, but others will require our ring and/or element to have additional properties. For example, we can only take $x^0 = 1$ if our ring has a multiplicative identity. Also, $x^{-n} = \left(x^{-1}\right)^n$ only makes sense when x is a unit (otherwise, x^{-1} does not exist!).

Definition 8.4 Let R be a ring, and let n be a positive integer.

- For all $x \in R$, we define $0x = 0_R$ and $(-n)x = n(-x)$, where $-x$ is the additive inverse of x.
- If R is a ring with identity, then for each nonzero $x \in R$, we define $x^0 = 1_R$. If R does not have identity, then x^0 remains undefined.
- If R is a ring with identity, then for each unit $x \in R$, we define $x^{-n} = (x{-}1)^n$, where x^{-1} denotes the multiplicative inverse of x. If R does not have identity, or if x is not a unit in R, then x^{-n} remains undefined.

Note: If n is a negative integer, then Definition 8.4 implies that $nx = (-n)(-x)$ and $x^n = \left(x^{-1}\right)^{-n}$, where $-n > 0$. These methods of expressing elements will be useful when proving later results in this section.

Theorem 8.5 Let R be a ring and let $x, y \in R$, and let m and n be integers. Then

i). $m(x + y) = mx + my$.

ii). $-(mx) = m(-x) = (-m)x$

iii). $(m + n)x = mx + nx$

iv). $m(nx) = (mn)x$

v). $m(xy) = (mx)y = x(my)$

vi). $(mx)(ny) = mn(xy)$.

1. The goal for this activity is to outline a proof for Theorem 8.5 part i).

   (a) Briefly explain why parts i) and iii) of Theorem 8.5 are not an immediate consequence of our Ring Axioms.

   (b) Our strategy for proving Theorem 8.5 part i) will be to consider three cases: $m = 0$, $m > 0$, and $m < 0$. Prove the first case, when $m = 0$.

(c) Now let $m > 0$, and let $P(m)$ be the statement "$m(x+y) = mx + my$". Prove that $P(1)$ is true.

(d) Let k be a positive integer, and assume that $P(k)$ is true. Use this to show that $P(k+1)$ is true (note that we often use inductive proofs to verify theorems involving recursive definitions). Use this and the previous part to conclude that the result is true for all $m \geq 0$

(e) Fill in the missing details for the proof outline on page 95 in your book to prove the case when $m < 0$.

Theorem 8.7: Let R be ring, let $x, y \in R$, and let m and n be positive integers. Then:

- $x^{m+n} = x^m x^n$
- $(x^m)^n = x^{mn}$

Note: If R is a ring with identity, then the properties in the Theorem above hold for all non-negative integers m and n (provided that $x \neq 0$ since we have not defined 0^0). Furthermore, if R is a ring with identity and x is a unit in R, then the above properties hold for all integers m and n.

2. The proof for each part of Theorem 8.7 are presentation problems (either for today or in class on Wednesday).

Lemma 8.8 Let R be a ring with identity, and let x be a unit in R. Then:

(a) The element x^{-1} is a unit in R and $\left(x^{-1}\right)^{-1} = x$.
(b) For every integer n, $x^n = x \cdot x^{n-1}$.
(c) For every integer n, x^n is a unit and $(x^n)^{-1} = \left(x^{-1}\right)^n$.

3. Prove Lemma 8.8 part (a).

4. Fill in the missing details to complete the proof of part (b) given on p. 98 of your textbook.