Math 476 - Abstract Algebra 1
Day 29 Group Assignment **Name:**________________________________

The Characteristic of a Ring; Subrings

1. Consider a standard 12-hour clock.

   (a) Suppose the clock reads 7:00. What time will it be in 6 hours? in 12 hours? in 18 hours? in 24 hours?

   (b) Describe the set of all positive integers k that make the statement "No matter what time is is currently, it will be the same time k hours from now" a true statement.

   (c) For which positive integers k does $k[3] = [0]$ in $\mathbb{Z}_{12}$? For which does $k[4] = [0]$? How about $k[5] = [0]$?

   (d) For which positive integers k does $k[x] = [0]$ for *all* $[x]$ in $\mathbb{Z}_{12}$? What it the smallest of these k values?

   (e) Is there a positive integer k such that $kx = 0$ for all $x \in \mathbb{Z}$? Why or why not?

Definition 8.11 Let R be a ring. The **characteristic** of R, denoted $char(R)$, is the smallest positive integer k such that $kx = 0_R$ for all $x \in R$. If not such integer exists, then R is said to have **characteristic zero**.

Theorem 8.12:

- For every integer $n \geq 2$, $char(\mathbb{Z}_n) = n$.
- The rings $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ all have characteristic zero.

Proof: Let $n \geq 2$ be given. In order to show that $char(\mathbb{Z}_n) = n$, we must show the following:

- that n is a positive integer;
- that $n[x] = [0]$ for all $[x] \in \mathbb{Z}_n$; and
- that there is no positive integer $k < n$ for which $k[x] = [0]$ for all $x \in \mathbb{Z}_n$.

The fact that n is positive is immediate, since $n \geq 2$ is given. To establish the second, notice that $n[x] = [nx] = [n][x] = [0][x] = [0]$. Note that we are taking advantage of the fact that n is an integer here.

Finally, let k be any positive integer with $k < n$. Then $n \not| k$. Therefore, $k[1] = [k] \neq [0]$. Hence $char(\mathbb{Z}_n) = n$. $\square$.

2. Is it possible for a ring to have characteristic 1? If so, given an example. If not, explain why it is not possible.

3. Let R be a ring and suppose $x, y \in R$. Is the statement $(xy)^m = x^m y^m$ true for all rings, some rings, or no rings? Justify your answer.

Definition 9.2 Let R be a ring and let S be a subset of R. Then S is said to be a **subring** of R provided that S is a ring with the operations of addition and multiplication defined the same as in R.

Theorem 9.3: Let R be a ring, and let S be a subset of R. Then S is a subring of R if:

- S is closed under addition
- S is closed under multiplication
- S contains 0_R (i.e. $0_R \in S$).
- S is closed under additive inverses (i.e. for every $x \in S$, $-x \in S$).

Theorem 9.4 (Subring Test): Let R be a ring, and let S be a subset of R. Then S is a subring of R if and only if

- S is non-empty
- S is closed under multiplication
- S is closed under subtraction

Proof: see p. 108 in your textbook.

4. Determine whether or not $\mathbb{E}$ is a subring of $\mathbb{Z}$.

5. Determine whether or not $K = \left\{ \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$ is a subring of $\mathcal{M}_{2\times 2}(\mathbb{R})$.

6. Determine whether or not the set $2\mathbb{Z}_6$ is a subring of $\mathbb{Z}_6$.

7. If time allows, you may present portions of Activity 9.7 on page 109 in your textbook. However, you must present (a), (b), and (c) at the same time, the same goes for (d) and (e); (f) and (g); (h) and (i); and (j) and (k).