 Name:______________________________________

Proving and Disproving Isomorphism

To show that two rings R and S **are** isomorphic, what we typically do is define a function $\varphi : R \to S$, prove that the function is well defined, prove that it is a bijection, and then prove that it preserves both the addition and multiplication operations.

To show that two rings are **not** isomorphic, we must prove that the does not exist a function $\varphi : R \to S$ that is an isomorphism. At first, this may seem to be a difficult task. Certainly, showing this directly is challenging (if not impossible). Instead of showing the impossibility of the existence of an isomorphism directly, it is more practical to use invariants to demonstrate the impossibility (usually in the context of a proof by contradiction). In the following activity, we will walk through an example.

1. Suppose that $\varphi : \mathbb{C} \to \mathbb{R}$ is a ring isomorphism.

   (a) Prove that $\varphi(0) = 0$.

   (b) Prove that $\varphi(1) = 1$.

   (c) Prove that $(\varphi(i))^2 = -1$ [Hint: Apply φ to both sides of the equation $i^2 + 1 = 0$].

   (d) Explain why the result you obtained in part (c) is a contradiction, allowing us to conclude that $\mathbb{C}$ and $\mathbb{R}$ are not isomorphic as rings.

Theorem 10.13 Let R and S be rings, and let $\varphi : R \to S$ be an isomorphism. If R has an identity 1_R, then S has an identity. Specifically, $\varphi(1_R)$ is an identity for S.

2. Give a complete proof of Theorem 10.13. [Presentation Eligible]

Corollary 10.15 The existence of an identity is an invariant of ring isomorphism. Specifically, if R and S are isomorphic rings and R has identity, then S also has identity.

3. In the space provided below, list the invariants of ring isomorphism given in your book on page 132.

   A Partial List of Invariants of Ring Isomorphism

4. Prove that commutativity is an invariant of ring isomorphism. [Presentation Eligible]

5. Complete each of the following using the rules for addition and multiplication of polynomials that you learned in high school algebra.

   (a) Let $f(x) = 2x^2 + 3x + 2$ and $g(x) = 3x^2 + 4x + 2$. Find $f(x) + g(x)$ and $f(x)g(x)$.

   (b) $f(x) = [2]x^2 + [3]x + [2]$ and $g(x) = [3]x^2 + [4]x + [2]$, where the coefficients are elements of $\mathbb{Z}_5$. Find $f(x) + g(x)$ and $f(x)g(x)$.

   (c) $f(x) = [2]x^2 + [3]x + [2]$ and $g(x) = [3]x^2 + [4]x + [2]$, where the coefficients are elements of $\mathbb{Z}_6$. Find $f(x) + g(x)$ and $f(x)g(x)$.